Laguerre polynomials

The Laguerre polynomial $L_k(x)$ is a polynomial of degree k that satisfies Laguerre's equation

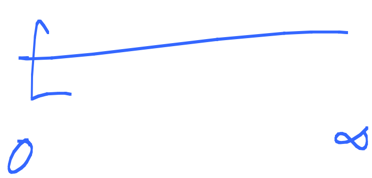
$$x y'' + (1-x) y' + k y = 0$$

which can be written in Sturm-Lioville form

as:

$$-(x e^{-x} y')' = k e^{-x} y$$

$p(x) = x e^{-x}$, $q = 0$, $\rho(x) = e^{-x}$, $\lambda = k$ $\quad \boxed{x \geq 0}$



$$L_k(x) = \frac{1}{n!} e^x \left(\frac{d}{dx}\right)^k \left(x^k e^{-x}\right)$$

Rodrigues' formula

Orthogonality

$$\int_0^\infty L_k(x) L_m(x) e^{-x} dx = 0 \qquad k \neq m$$

Normalization

$$\int_0^\infty (L_k(x))^2 e^{-x} dx = 1$$

$$f(x) = \sum_{k=0}^{\infty} a_k L_k(x)$$

Expansion

$$a_k = \int_0^\infty f(x) L_k(x) e^{-x} dx$$

Generating function

$$\frac{1}{1-u}\, e^{-\frac{xu}{1-u}} = \sum_{k=0}^{\infty} L_k(x)\, u^k$$

Recurrence relations

$$L_k(x) = L_k'(x) - L_{k+1}'(x)$$

$$(k+1) L_{k+1}(x) - (2k+1-x) L_k(x) + k L_{k-1}(x) = 0$$

$$x L_k'(x) = k\left(L_k(x) - L_{k-1}(x)\right)$$

$$L_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{x^j}{j!}$$

Associated Laguerre polynomials

$$L_k^m(x) = \sum_{j=0}^{k} (-1)^j \binom{k+m}{j+m} \frac{x^j}{j!}$$

or by Rodrigues' formula

$$L_k^m(x) = \frac{1}{k!} e^x x^{-m} \left(\frac{d}{dx}\right)^k \left(x^{k+m} e^{-x}\right)$$

or

$$L_k^m(x) = (-1)^m \left(\frac{d}{dx}\right)^m \left(L_{k+m}(x)\right)$$

L_k^m satisfies the modified Laguerre's equation

$$x y'' + (m+1-x) y' + k y = 0$$

Orthogonality

$$\int_0^\infty L_k^m L_j^m \, x^m e^{-x} dx = 0 \quad k \neq j$$

Normalization

$$\int_0^\infty (L_k^m)^2 \, x^m e^{-x} dx = \frac{(k+m)!}{k!}$$

Generating function

$$\frac{1}{(1-u)^{m+1}} \, e^{-\frac{xu}{1-u}}$$

Recurrence relations

$$x L_k^{m\,\prime}(x) = k L_k^m(x) - (k+m) L_{k-1}^m(x)$$

$$(k+1) L_{k+1}^m - (2k+m+1-x) L_k^m - (k+m) L_{k-1}^m = 0$$