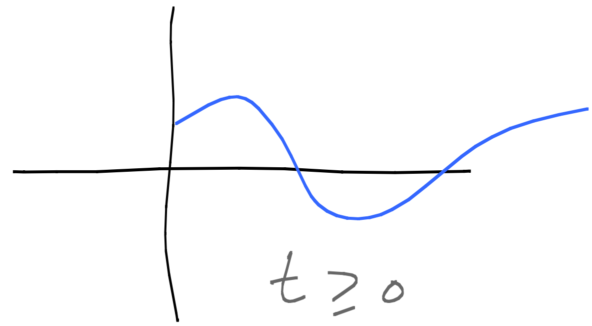
Laplace Transforms

$$f : [0,\infty) \longrightarrow \mathbb{R}$$
$$t \longmapsto f(t)$$



$\mathcal{L}(f)(s) = F(s)$ is defined by:

$$\boxed{F(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$

(converges for $s \geq s_0$ for some s_0)

s can assume complex values.

Fundamental Properties:

0. $\mathcal{L}$ is linear:

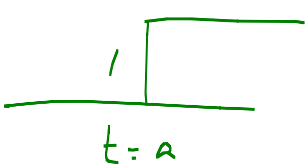
$$\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2)$$

1. Translations:

$$\mathcal{L}(e^{-at} f(t)) = F(s+a)$$

$$\mathcal{L}(H(t-a) f(t-a)) = e^{-as} F(s)$$

Heaviside



2. Dilation:

$$\mathcal{L}\left(f\left(\frac{t}{\lambda}\right)\right) = \lambda F(\lambda s) \qquad \lambda > 0$$

3. Differentiation:

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

and more generally:

$$\mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$$

$$\mathcal{L}(t f(t)) = F'(s)$$

4. Integration

$$\mathcal{L}\left(\int_0^t f\right) = \frac{1}{s} F(s)$$

5. Convolution

$$\mathcal{L}(f_1 * f_2) = F_1(s)\,F_2(s)$$

where

$$f_1 * f_2(t) = \int_0^t f_1(u)\,f_2(t-u)\,du$$

Basic Examples:

$f(t)$	$F(s)$
$\delta(t-a)$ Dirac	e^{-as}
$H(t-a)$ Heaviside	$\frac{1}{s}e^{-as}$
1 constant	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
t^α, $\alpha > -1$	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ Gamma function

$$e^{\lambda t} \qquad \frac{1}{s-\lambda}$$

$$\sin(\omega t) \qquad \frac{\omega}{s^2+\omega^2} \qquad \frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right)$$

$$\cos(\omega t) \qquad \frac{s}{s^2+\omega^2} \qquad \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)$$

$$\sinh(\omega t) \qquad \frac{\omega}{s^2-\omega^2} \qquad \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right)$$

$$\cosh(\omega t) \qquad \frac{s}{s^2-\omega^2} \qquad \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right)$$

$$t^n e^{\lambda t} \qquad \frac{n!}{(s-\lambda)^{n+1}}$$

Example of solving a simple ODE

15.10 (a)

$$f''(t) + 5f'(t) + 6f(t) = 0$$

$$f(0) = 1,\ f'(0) = 4$$

$\mathcal{L}$

$$(s^2F(s) - sf(0) - f'(0)) + 5(sF(s) - f(0)) + 6F(s) = 0$$

$$(s^2 + 5s + 6)\ F(s) = s + 1$$

$$F(s) = \frac{s+1}{(s+2)(s+3)} = \frac{2}{s+3} - \frac{1}{s+2}$$

$\mathcal{L}^{-1}$

$$f(t) = 2e^{-3t} - e^{-2t}$$

solution

Fourier transforms can also be used to solve O.D.E.'s

13.5 $$\phi''(x) - \omega^2 \phi(x) = g(x)$$

$$\mathcal{F} \Big\downarrow$$

$$-k^2 \hat{\phi}(k) - \omega^2 \hat{\phi}(k) = \hat{g}(k)$$

$(+ik)^2 = -k^2$

$$\hat{\phi}(k) = \frac{-\hat{g}(k)}{k^2 + \omega^2}$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$\phi(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(k)\, e^{ikx}}{k^2 + \omega^2}\, dx$$ solution