Notes on Face-listings for

"Classification of Superpotentials" by A. Dancer and M. Wang

These notes provide a listing of possible 2-dimensional faces of conv(W), where W is the set of weight vectors for the scalar curvature function of a compact homogeneous space G/K where the isotropy representation is multiplicity free. While they may be of independent interest, these notes are mainly intended as a supplement to section 6 of our paper "Classification of Superpotentials". Accordingly, we shall use the notation and definitions given there. In particular, these notes also list the possible triangles that can occur in Theorem 6.12 of the above paper.

GENERAL REMARKS

We proceed by studying all possible triples of vectors in W whose affine span is a 2-plane. In most cases there are no further vectors in this plane, so we obtain a triangle. In some cases there are further vectors in the 2-plane; it turns out that the full set of possible vectors in the 2-plane may give a hexagon, square, trapezium or parallelogram.

For each possible triangle (including subtriangles of the hexagons, parallelograms etc) we test whether it may give a triangle as in Thm 6.12(i). We list the possible vectors

```
x''    x     x'       3c     3a    3a'
```

and test whether it satisfies the conditions of Thm 6.12(i), i.e., that c,a,a' are all null and that x'' is orthogonal to a,a'. The cases where this condition can hold give examples (Tr1)-(Tr22).

We recall that if x''xx' gives a triangle satisfying the conditions of Thm 6.12(i) then x'' cannot be type I. Moreover if x'' is type III, say (-2^i, 1^j), then x_i = x'_i iff x_i = x'_j.

In some cases the shape in the 2-plane cannot be a face. In some of these cases it is possible that a subtriangle may still be a face, so for these we still have to check if the conditions of Thm 6.12(i) can hold.

We note that configurations involving the following (column) vectors will never give a face besides case 0) (see below):

```
-2    1        or      1  -1
 1   -1               -1
     -1               -1
```

This is because the face condition forces a spanning set for the 2-plane in 0) to lie in the given 2-plane. So if the latter has a further vector not in 0) we arrive at a contradiction.

Also if the configuration

```
-1
 1     -1
-1      1
       -1
```

occurs, then the face condition implies that

```
-1
-1      1
 1     -1
       -1
```

also lies in the 2-plane. Many cases can be eliminated by this observation.

Recall from Remarks 6.13, 6.14 that no triangle containing points of W in the interior of an edge can satisfy the conditions of Thm 6.12. Hence we do not need to treat such triangles.

We frequently make use without comment of symmetries in the configuration to reduce the number of cases that need be checked.

Finally, we also check which triangles can satisfy the conditions of Theorem 6.12(ii). These conditions are symmetric with respect to x'', x, x'. Recall that now one vector, say x'', must be type I. Moreover c,a,a' must be null. Writing x'' = (-1^i), we find that nullity of a,a' implies x_i = x'_i.

CONTENTS

The different cases can be grouped according to the types of the vectors in a spanning set.

0) to 15)	Three type III
16) to 22)	Two type III and a type I
23) to 79)	Two type III and a type II
80) to 90a)	A type III, a type II and a type I
91) to 154)	A type III and two type II
155)	Two type I and a type II
156) to 177)	Two type II and one type I
178) to end	Three type II vectors

Note that the case of three type I is included in 0). Also, the case of a type III and two type I is dealt with in the comment after 90a).

0) If all three vectors are zero outside a common set of three indices, we have the hexagon (H1) lying in the 2-plane

X_1 + X_2 + X_3 =-1   X_i =0 for i > 3.

The only way to get subtriangles of the hexagon with no interior points of edges is by taking the three type I vectors.

x''	x	x'	3c	3a	3a'	
-1	0	0	1	-1	-1	a,c not both null
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	

****

In future, therefore, we need only consider triples which between them involve nonzero entries in more than three places.

We first consider 2-planes including three type III vectors.

****

1) triangle

x''	x	x'	3c	3a	3a'	
-2	-2	-2	-6	-6	-6	
1	0	0	-1	1	1	a,c not both null
0	1	0	2	4	-2	
0	0	1	2	-2	4	

****

2)

-2	-2	0
1	0	1
0	1	0
0	0	-2

The 2-plane is given by

X_1 + X_4 =-2, X_2 + X_3 =1, X_i =0 : i > 4

and contains in addition the vectors

0	-1	-1
0	1	0
1	0	1
-2	-1	-1

This is the rectangle (P17). We must consider subtriangles

x''	x	x'	3c	3a	3a'	
-2	-2	-1	-4	-8	-2	
1	0	1	1	-1	5	a,c not both null
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	
-2	-2	0	-2	-10	2	
1	0	1	1	-1	5	ditto
0	1	0	2	4	-2	
0	0	-2	-4	4	-8	

-2	-1	0	0	-6	0	
1	0	1	1	-1	5	
0	1	0	2	4	-2	a',c not both null
0	-1	-2	-6	0	-6	

-2	-1	0	0	-6	0	
1	1	0	1	5	-1	ditto
0	0	1	2	-2	4	
0	-1	-2	-6	0	-6	

-1	-2	-1	-5	-7	-1	orthogonality implies
1	0	0	-1	1	1	$(d_1,d_2)=(3,1)$ so a
0	1	1	4	2	2	not null
-1	0	-1	-1	1	-5	

-1	-2	0	-3	-9	3	as above
1	0	0	-1	1	1	
0	1	1	4	2	2	
-1	0	-2	-3	3	-9	

-1	-2	-2	-7	-5	-5	orthogonality
1	1	0	1	5	-1	conditions cannot
0	0	1	2	-2	4	both hold
-1	0	0	1	-1	-1	

-1	-2	-1	-5	-7	-1	
1	1	0	1	5	-1	a,c not both null
0	0	1	2	-2	4	
-1	0	-1	-1	1	-5	

-1	-2	0	-3	-9	3	
1	1	0	1	5	-1	ditto
0	0	1	2	-2	4	
-1	0	-2	-3	3	-9	

****

3)

-2	-2	0	now	0	0	also lie in the 2-plane
1	0	-2		0	-1	
0	1	0		-2	-1	
0	0	1		1	1	

$X_2 + X_3 + 3 X_4 = 1$, $X_1 + X_2 + X_3 + X_4 = -1$, $X_i = 0 : I > 4$.

This is the trapezium (T1).

we must consider subtriangles

0	-2	0	-4	-8	4	
-2	0	-1	0	0	-6	
0	1	-1	0	6	-6	a,c not both null
1	0	1	1	-1	5	

0	-2	0	-4	-8	4	
-2	1	0	4	2	-4	a',c not both null
0	0	-2	-4	4	-8	
1	0	1	1	-1	5	

0	-2	0	-4	-8	4	
-2	1	-1	2	4	-8	
0	0	-1	-2	2	-4	ditto
1	0	1	1	-1	5	

-2	-2	0	-2	-10	2	
0	1	-1	0	6	-6	
1	0	-1	-3	3	-3	a,c not both null
0	0	1	2	-2	4	

-2	-2	0	-2	-10	2	
0	1	0	2	4	-2	ditto
1	0	-2	-5	5	-7	
0	0	1	2	-2	4	

0	-2	-2	-8	-4	-4	
-1	0	1	3	-3	3	
-1	1	0	3	3	-3	ditto
1	0	0	-1	1	1	

0	0	-2	-4	4	-8	
-1	-2	0	-3	-9	3	ditto
-1	0	1	3	-3	3	
1	1	0	1	5	-1	

0	0	-2	-4	4	-8	
-1	-2	1	-1	-11	-7	ditto
-1	0	0	1	-1	-1	
1	1	0	1	-5	-1	

****

4)triangle	-2	0	0	2	-2	-2	
	1	-2	-2	-9	-3	-3	orthogonality fails as
	0	1	0	2	4	-2	d_1 is not 1
	0	0	1	2	-2	4	

****

5)triangle	-2	0	0	2	-2	-2	
	1	0	0	-1	1	1	
	0	-2	-2	-8	-4	-4	This is (Tr1)
	0	1	0	2	4	-2	
	0	0	1	2	-2	4	

****

6)triangle

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	-2	0	-4	-8	4	a,c not both null
0	1	0	2	4	-2	
0	0	-2	-4	4	-8	
0	0	1	2	-2	4	

****

7)

-2	1	0		-1	0	
1	-2	0	now	0	-1	also occur.
0	0	-2		0	0	
0	0	1		0	0	

This is a triangle with two interior points on one side. By Remark 6.13, 6.14 the subtriangles to consider are

-2	0	-1	0	0	-6	
1	0	0	-1	1	1	a', c not both null
0	-2	0	-4	-8	4	
0	1	0	2	4	-2	

0	-2	-1	-6	-6	0	
0	1	0	2	4	-2	
-2	0	0	2	-2	-2	ditto
1	0	0	-1	1	1	

0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	ditto
-2	0	0	2	-2	-2	
1	0	0	-1	1	1	

****

8) triangle

0	-2	0	-4	-8	4	
-2	1	0	4	2	-4	a', c not both null
1	0	-2	-5	5	-7	
0	0	1	2	-2	4	

****

9)

-2	0	0	
1	1	0	equivalent to 3)
0	-2	1	
0	0	-2	

****

10)

-2	0	1	
1	-2	0	equivalent to 8)
0	1	0	
0	0	-2	

****

11)

-2	0	0		-1	
1	-2	1	now	1	is present.
0	0	-2		0	
0	1	0		-1	

This is a triangle with a midpoint of one side. Subtriangles are

-2	0	-1	0	0	-6	
1	-2	1	-3	-9	9	a, c not both null
0	0	-1	-2	2	-4	
0	1	0	2	4	-2	

-1	0	-2	-3	3	-9	
1	-2	1	-3	-9	9	ditto
-1	0	0	1	-1	-1	
0	1	0	2	4	-2	

0	-2	0	-4	-8	4	
-2	1	1	6	0	0	
0	0	-2	-4	4	-8	orthogonality fails
1	0	0	-1	1	1	

0	-2	-1	-6	-6	0	
-2	1	1	6	0	0	
0	0	-1	-2	2	-4	a,c not both null
1	0	0	-1	1	1	

****

12)	-2	0	0		-1	0	-1
	1	1	1	Now	1	1	1
	0	-2	0		0	-1	-1
	0	0	-2		-1	-1	0

are also in the 2-plane X_2 =1 : X_i =0 for i > 4. This is a triangle with midpoints of all sides.

Subtriangles are

-2	0	0	2	-2	-2	
1	1	1	3	3	3	
0	-2	0	-4	-8	4	a,c not both null
0	0	-2	-4	4	-8	

-2	0	0	2	-2	-2	
1	1	1	3	3	3	
0	-2	-1	-6	-6	0	this is (Tr2)
0	0	-1	-2	2	-4	

-2	-1	-1	-2	-4	-4	
1	1	1	3	3	3	a,c not both null
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
1	1	1	3	3	3	ditto
-1	0	-2	-3	3	-9	
-1	0	0	1	-1	-1	

-1	-2	-1	-5	-7	-1	
1	1	1	3	3	3	ditto
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	

-1	-2	0	-3	-9	3	
1	1	1	3	3	3	ditto
-1	0	-1	-1	1	-5	
0	0	-1	-2	2	-4	

0	-2	-1	-6	-6	0	
1	1	1	3	3	3	
-1	0	-1	-1	1	-5	orthogonality fails
-1	0	0	1	-1	-1	

-1	0	-1	-1	1	-5	nullity and
1	1	1	3	3	3	othogonality are
-1	-1	0	-1	-5	1	contradictory
0	-1	-1	-4	-2	-2	

****

13) triangle

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	-2	0	-4	-8	4	
0	1	-2	-2	8	-10	a,c not both null
0	0	1	2	-2	4	

****

14)

-2	0	0	now	-1 is present.
1	1	0		1
0	-2	0		-1
0	0	-2		0
0	0	1		0

This is a triangle with midpoint of one edge.

-2	0	-1	0	0	-6	
1	0	1	1	-1	5	
0	0	-1	-2	2	-4	a,c not both null
0	-2	0	-4	-8	4	
0	1	0	2	4	-2	

-2	0	0	2	-2	-2	
1	0	1	1	-1	5	
0	0	-2	-4	4	-8	ditto
0	-2	0	-4	-8	4	
0	1	0	2	4	-2	

-1	0	-2	-3	3	-9	
1	0	1	1	-1	5	
-1	0	0	1	-1	-1	ditto
0	-2	0	-4	-8	4	
0	1	0	2	4	-2	

0	-2	-1	-6	-6	0	
0	1	1	4	2	2	
0	0	-1	-2	2	-4	ditto
-2	0	0	2	-2	-2	
1	0	0	-1	1	1	

0	-2	0	-4	-8	4	
0	1	1	4	2	2	
0	0	-2	-4	4	-8	this is (Tr3)
-2	0	0	2	-2	-2	

```
                1       0      0       -1       1      1

***************

15)         -2  0  0
             1  0  0
             0 -2  1     equivalent to 13)
             0  1  0
             0  0 -2

***************
```

Now we consider two type III and one type I.

```
16)    -2  -2   0   now  0   0 are present in the 2-plane
        1   0   0        1  -1
        0   1   0       -1   1
        0   0  -1       -1  -1
```

given by X_1 + X_2 + X_3 + X_4 = -1, X_2 + X_3 - X_4 = 1, X_i = 0: i > 4.

This is trapezium (T2). The type I must be present; if one type II is present the other is. So the only way to obtain a triangle without midpoints is to consider the type I and the two type III. This is ruled out by Remark 6.15

****

17) triangle

```
                0    -2     0      -4    -8     4
               -2     1     0       4     2    -4     a',c not both null
                1     0     0      -1     1     1
                0     0    -1      -2     2    -4

***************

18)     -2  0  0
         1  0  0
         0 -2 -1     equivalent to 7)
         0  1  0
         0  0  0

***************
19)     -2  0  0
         1  0  0
         0 -2  0     equivalent to 7)
         0  1 -1
         0  0  0

***************

20)   -2  0  0       Now  -1  1 -1  are also in the 2-plane
       1  1  0             0  0  1
       0 -2  0             1 -1 -1
       0  0 -1            -1 -1  0
```

```
X_1 + 2X_2 + X_3 = 0, X_2 - X_4 =1, X_i =0 for i>4.
```

This is the parallelogram (P16). Subtriangles to consider are

```
-2     0    -1     0     0    -6
 1     0     1     1    -1     5        a,c not both null
 0     0    -1    -2     2    -4
 0    -1     0    -2    -4     2

-1     0    -2    -3     3    -9
 1     0     1     1    -1     5           ditto
-1     0     0     1    -1    -1
 0    -1     0    -2    -4     2
```

****

```
21) triangle   -2  0  0
                1  0  0
                0 -2  0
                0  1  0
                0  0 -1


               -2     0     0     2    -2    -2
                1     0     0    -1     1     1
                0    -2     0    -4    -8     4      a,c not both null
                0     1     0     2     4    -2
                0     0    -1    -2     2    -4
```

****

If there are two type I in the 2-plane then there are two type III. So we do not need to consider type III and two type I further.

For example

```
22)      -2  0  0
          1  0  0   is equivalent to 7)
          0 -1  0
          0  0 -1
```

****

Next we consider 2-planes including two type III and a type II.

```
Observe that any 2-plane including -2  and  -1 will also include  0
                                     1        1                    1
                                     0       -1                   -2
```

so we do not consider further examples with two type III and a type II of the above form.

For example:

```
23)      -2 -2   -1  or -1
          1  0    0      1   is equivalent to 2).
          0  1    1      0
          0  0   -1     -1
```

****

```
24)      -2  -2   -1    now  -1  is in the plane
          1   0    0         -1
          0   1   -1          0
          0   0    1          1
```

X_2 + X_3 + 2X_4 =1, X_1 + X_2 + X_3 + X_4 = -1, X_i =0 : i > 4.
This is parallelogram (P1). Subtriangles to consider are:

```
        -2     -2    -1      -4     -8    -2
         1      0    -1      -3      3    -3   a,c not both null
         0      1     0       2      4    -2
         0      0     1       2     -2     4

        -1     -2    -2      -7     -5    -5
         0      1     0       2      4    -2   a', c not both null
        -1      0     1       3     -3     3
         1      0     0      -1      1     1

        -1     -2    -1      -5     -7    -1
         0      1    -1       0      6    -6   a, c not both null
        -1      0     0       1     -1    -1
         1      0     1       1     -1    -5

        -1     -2    -1      -5     -7    -1
         0      0    -1      -2      2    -4         ditto
        -1      1     0       3      3    -3
         1      0     1       1     -1     5
```

****

```
25) triangle
        -1     -2    -2      -7     -5    -5
         0      1     0       2     -2     4  orthogonality implies
         0      0     1       2      4    -2  d_1 \leq 5,  which
         1      0     0      -1      1     1  contradicts nullity of c
        -1      0     0       1     -1    -1
```

****

```
26)     -2  -2   1   now   1  is also present.
         1   0   0        -1
         0   1  -1         0
         0   0  -1        -1
```

This is a parallelogram, but not a face by remark ..

```
        -2     -2     1       0    -12     6
         1      0    -1      -3      3    -3
         0      1     0       2      4    -2    a,c not both null
         0      0    -1      -2      2    -4
```

1	-2	1	-3	-9	9	
0	0	-1	-2	2	-4	
-1	1	0	3	3	-3	ditto
-1	0	-1	-1	1	-5	

1	-2	-2	-9	-3	-3	
0	0	1	2	-2	4	ditto
-1	1	0	3	3	-3	
-1	0	0	1	-1	-1	

1	1	-2	-3	9	-9	
0	-1	1	0	-6	6	ditto
-1	0	0	1	-1	-1	
-1	-1	0	-1	-5	1	

****

27) triangle

1	-2	-2	-9	-3	-3	
0	1	0	2	4	-2	
0	0	1	2	-2	4	orthogonality fails
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

28)

-2	-2	0	Now	0	is in 2-plane; equivalent to 3).
1	0	-1		0	
0	1	-1		-2	
0	0	1		1	

****

29)

-2	-2	0	
1	0	1	This is equivalent to 16).
0	1	-1	
0	0	-1	

****

30)

-2	-2	0	Now	0	is also present.
1	0	0		1	
0	1	1		0	
0	0	-1		-1	
0	0	-1		-1	

This is parallelogram (P2). Subtriangles to consider are

-2	-2	0	-2	-10	2	
1	0	1	1	-1	5	
0	1	0	2	4	-2	a,c not both null
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
0	0	1	2	-2	4	
1	1	0	1	5	-1	ditto
-1	0	-1	-1	1	-5	
-1	0	-1	-1	1	-5	

0	-2	0	-4	-8	4	
0	1	1	4	2	2	
1	0	0	-1	1	1	orthogonality fails
-1	0	-1	-1	1	-5	
-1	0	-1	-1	1	-5	

0	-2	-2	-8	-4	-4	
0	1	0	2	4	-2	
1	0	1	1	-1	5	ditto
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

31)	-2	-2	0	Now	0 is in the 2-plane
	1	0	-1		0
	0	1	0		-1
	0	0	-1		-1
	0	0	1		1

This is parallelogram (P3). Subtriangles to consider are

-2	-2	0	-2	-10	2	
1	0	-1	-3	3	-3	
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	a,c not both null
0	0	1	2	-2	4	

0	-2	0	-4	-8	4	
0	0	-1	-2	2	-4	
-1	1	0	3	3	-3	ditto
-1	0	-1	-1	1	-5	
1	0	1	1	-1	5	

0	-2	-2	-8	-4	-4	
0	0	1	2	-2	4	ditto
-1	1	0	3	3	-3	
-1	0	0	1	-1	-1	
1	0	0	-1	1	1	

0	0	-2	-4	4	-8	
0	-1	1	0	-6	6	
-1	0	0	1	-1	-1	
-1	-1	0	-1	-5	1	ditto
1	1	0	1	5	-1	

****

32) triangle

0	-2	-2	-8	-4	-4	
0	1	0	2	4	-2	
0	0	1	2	-2	4	
1	0	0	-1	1	1	this is (Tr11)
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

```
33) -2   1   -1
     1  -2    0
     0   0   -1
     0   0    1
```

This lies in the 2-plane X_1 + X_2 =-1; X_3 + X_4 =0; X_i =0: i>4.

```
Now    -1  0 -1   are also in the 2-plane.
        0 -1  0
        1  0  0
       -1  0  0
```

This is hexagon H2. There is no way to get a subtriangle without midpoints.

****

```
34)      -2      1    1    Now  0  -1   0 are also in the 2-plane
          1     -2    0         1   0  -1
          0      0   -1        -1   0   0
          0      0   -1        -1   0   0
```

X_3 = X_4, X_1 + X_2 + X_3 + X_4 = -1, X_i =0: i > 4. This is the trapezium (T*1).

We must consider subtriangles

```
       0     -1     0      -2     -4     2
       1      0    -1      -3      3    -3      a,c not both null
      -1      0     0       1     -1    -1
      -1      0     0       1     -1    -1

       1     -2    -1      -7     -5     1
       0      1     0       2      4    -2      a, a' not both null
      -1      0     0       1     -1    -1
      -1      0     0       1     -1    -1
```

****

```
35)      -2   1   0    Now -1  0  are also in the 2-plane.
          1  -2   0         0 -1
          0   0  -1         0  0
          0   0  -1         0  0
          0   0   1         0  0
```

This is a triangle with 2 interior points on one side.
Subtriangles to consider are

```
      -2      0    -1       0      0    -6
       1      0     0      -1      1     1
       0     -1     0      -2     -4     2    a,c not both null
       0     -1     0      -2     -4     2
       0      1     0       2      4    -2
```

```
         0      -2     -1       -6      -6      0
         0       1      0        2       4     -2       ditto
        -1       0      0        1      -1     -1
        -1       0      0        1      -1     -1
         1       0      0       -1       1      1

         0      -2      0       -4      -8      4
         0       1     -1        0       6     -6
        -1       0      0        1      -1     -1       ditto
        -1       0      0        1      -1     -1
         1       0      0       -1       1      1

         0      -2      1       -2     -10      8
         0       1     -2       -2       8    -10
        -1       0      0        1      -1     -1       ditto
        -1       0      0        1      -1     -1
         1       0      0       -1       1      1

         0      -1      0       -2      -4      2
         0       0     -1       -2       2     -4
        -1       0      0        1      -1     -1       ditto
        -1       0      0        1      -1     -1
         1       0      0       -1       1      1
```

****

```
36)      -2  0 -1     equivalent to 12)
          1  1  1
          0 -2  0
          0  0 -1
```

****

```
37)      -2  0 -1     equivalent to 20)
          1  1  0
          0 -2  1
          0  0 -1
```

****

```
38)      -2  0  1    now  0  -1  are in the same 2-plane
          1  1 -1        -1   1
          0 -2  0         1  -1
          0  0 -1        -1   0
```

This is the trapezium (T*2). Subtriangles to consider are

```
        -2       0     -1        0       0     -6
         1      -1      1       -1      -5      7    a,c not both null
         0       1     -1        0       6     -6
         0      -1      0       -2      -4      2

         0      -2     -1       -6      -6      0
        -1       1      1        5       1      1        ditto
         1       0     -1       -3       3     -3
        -1       0      0        1      -1     -1
```

0	1	-1	0	6	-6	
-1	-1	1	1	-7	5	ditto
1	0	-1	-3	3	-3	
-1	-1	0	-1	-5	1	

-1	0	1	3	-3	3	
1	-1	-1	-5	-1	-1	ditto
-1	1	0	3	3	-3	
0	-1	-1	-4	-2	-2	

-1	-2	0	-3	-9	3	
1	1	-1	-1	7	-5	ditto
-1	0	1	3	-3	3	
0	0	-1	-2	2	-4	

****

39)	-2	0	-1	equivalent to 2)
	1	1	0	
	0	-2	-1	
	0	0	1	

****

40)	-2	0	-1	Now	-1	0	are also in the 2-plane
	1	1	-1		1	-1	
	0	-2	0		-1	-1	
	0	0	1		0	1	

$X_2 + 2 X_4 = 1$, $X_1 + X_2 + X_3 + X_4 = -1$, $X_i = 0 : i > 4$.

This is trapezium (T3). Subtriangles to consider are

-2	-1	0	0	-6	0	
1	1	-1	-1	7	-5	
0	-1	-1	-4	-2	-2	orthogonality fails
0	0	1	2	-2	4	

-2	0	-1	0	0	-6	
1	1	-1	-1	7	-5	
0	-2	0	-4	-8	4	a,c not both null
0	0	1	2	-2	4	

-1	0	-2	-3	3	-9	
-1	-1	1	1	-7	5	ditto
0	-1	0	-2	-4	2	
1	1	0	1	5	-1	

-1	0	-1	-1	1	-5	orthogonality implies
-1	-1	1	1	-7	5	$d_1 = 1$, so a' is not
0	-1	-1	-4	-2	-2	null
1	1	0	1	5	-1	

-1	0	0	1	-1	-1	
-1	-1	1	1	-7	5	a', c not both null
0	-1	-2	-6	0	-6	
1	1	0	1	5	-1	

```
-1    -2     0     -3    -9     3
-1     1     1      5     1     1   orthogonality fails
 0     0    -2     -4     4    -8
 1     0     0     -1     1     1

-1    -2    -1     -5    -7    -1
-1     1     1      5     1     1       ditto
 0     0    -1     -2     2    -4
 1     0     0     -1     1     1

-1     0    -1     -1     1    -5
-1     1     1      5     1     1   a,c not both null
 0    -2    -1     -6    -6     0
 1     0     0     -1     1     1

-1    -1     0     -1    -5     1   orthogonality implies
 1    -1    -1     -5    -1    -1     d_1 =1, so a is not
-1     0    -1     -1     1    -5     null
 0     1     1      4     2     2

-1    -2    -1     -5    -7    -1
 1     1    -1     -1     7    -5   a,c not both null
-1     0     0      1    -1    -1
 0     0     1      2    -2     4

-1    -2     0     -3    -9     3
 1     1    -1     -1     7    -5
-1     0    -1     -1     1    -5        ditto
 0     0     1      2    -2     4
```

****

```
41)     -2  0  0       Now -1  is also in the 2-plane.
         1  1  1            1
         0 -2  0           -1
         0  0 -1            0
         0  0 -1            0
```

This is a triangle with midpoint of one edge present.

```
-2     0     0      2    -2    -2
 1     1     1      3     3     3
 0    -2     0     -4    -8     4   a,c not both null
 0     0    -1     -2     2    -4
 0     0    -1     -2     2    -4

-1     0    -2     -3     3    -9
 1     1     1      3     3     3
-1     0     0      1    -1    -1       ditto
 0    -1     0     -2    -4     2
 0    -1     0     -2    -4     2

 0    -2     0     -4    -8     4
 1     1     1      3     3     3
 0     0    -2     -4     4    -8       ditto
-1     0     0      1    -1    -1
```

-1	0	0	1	-1	-1	

0	-2	-1	-6	-6	0	
1	1	1	3	3	3	
0	0	-1	-2	2	-4	this is (Tr12)
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

42

-2	0	1	Now	0	-1
1	1	0		0	1
0	-2	0		1	-1
0	0	-1		-1	0
0	0	-1		-1	0

are in the 2-plane also,

given by X_4 = X_5, X_2 - X_5 = 1, X_1 + X_2 + X_3 + X_4 + X_5 =-1, X_i =0 :

i > 5. This is the trapezium (T4).

-2	0	-1	0	0	-6	
1	0	1	1	-1	5	
0	1	-1	0	6	-6	a,c not both null
0	-1	0	-2	-4	2	
0	-1	0	-2	-4	2	

-2	1	-1	2	4	-8	
1	0	1	1	-1	5	
0	0	-1	-2	2	-4	ditto
0	-1	0	-2	-4	2	
0	-1	0	-2	-4	2	

-2	1	0	4	2	-4	
1	0	1	1	-1	5	
0	0	-2	-4	4	-8	a', c not both null
0	-1	0	-2	-4	2	
0	-1	0	-2	-4	2	

0	1	-2	-2	8	-10	
0	0	1	2	-2	4	
1	0	0	-1	1	1	ditto
-1	-1	0	-1	-5	1	
-1	-1	0	-1	-5	1	

0	1	-1	0	6	-6	
0	0	1	2	-2	4	
1	0	-1	-3	3	-3	ditto
-1	-1	0	-1	-5	1	
-1	-1	0	-1	-5	1	

0	1	0	2	4	-2	
0	0	1	2	-2	4	
1	0	-2	-5	5	-7	ditto
-1	-1	0	-1	-5	1	
-1	-1	0	-1	-5	1	

```
                 0     -2     -1     -6     -6      0
                 0      1      1      4      2      2
                 1      0     -1     -3      3     -3        ditto
                -1      0      0      1     -1     -1
                -1      0      0      1     -1     -1

                 0     -2      0     -4     -8      4
                 0      1      1      4      2      2
                 1      0     -2     -5      5     -7    orthogonality fails
                -1      0      0      1     -1     -1
                -1      0      0      1     -1     -1

                 0     -1      0     -2     -4      2
                 0      1      1      4      2      2
                 1     -1     -2     -7      1     -5        ditto
                -1      0      0      1     -1     -1
                -1      0      0      1     -1     -1

                -1      0      1      3     -3      3
                 1      0      0     -1      1      1
                -1      1      0      3      3     -3    a,c not both null
                 0     -1     -1     -4     -2     -2
                 0     -1     -1     -4     -2     -2

                -1     -2      1     -1    -11      7
                 1      1      0      1      5     -1
                -1      0      0      1     -1     -1       ditto
                 0      0     -1     -2      2     -4
                 0      0     -1     -2      2     -4

                -1     -2      0     -3     -9      3
                 1      1      0      1      5     -1       ditto
                -1      0      1      3     -3      3
                 0      0     -1     -2      2     -4
                 0      0     -1     -2      2     -4

                                 ***************

43)       -2  0  -1   now   0 -1  are also in the 2-plane
           1  1   0         0  1
           0 -2   0        -1 -1
           0  0   1         1  0
           0  0  -1        -1  0

X_4 + X_5 =0, X_2 + X_4 =1, X_1 + X_2 + X_3 = -1, X_i = 0 : i > 5.
This is trapezium (T5). Subtriangles to consider are

          -2  -1   0    0  -6   0
           1   0   1    1  -1   5     a,c not both null
           0   0  -2   -4   4  -8
           0   1   0    2   4  -2
           0  -1   0   -2  -4   2

          -2   0  -1    0   0  -6
           1   0   1    1  -1   5
           0  -1  -1   -4  -2  -2     orthogonality fails
           0   1   0    2   4  -2
```

```
 0  -1   0   -2  -4   2

-1   0  -2   -3   3  -9
 0   0   1    2  -2   4     a,c not both null
 0  -1   0   -2  -4   2
 1   1   0    1   5  -1
-1  -1   0   -1  -5   1

-1   0  -1   -1   1  -5
 0   0   1    2  -2   4     orthogonality implies d_1 =1
 0  -1  -1   -4  -2  -2      so a is not null
 1   1   0    1   5  -1
-1  -1   0   -1  -5   1

-1   0   0    1  -1  -1
 0   0   1    2  -2   4     a', c not both null
 0  -1  -2   -6   0  -6
 1   1   0    1   5  -1
-1  -1   0   -1  -5   1

-1  -2  -1   -5  -7  -1
 0   1   1    4   2   2
 0   0  -1   -2   2  -4     orthogonality fails
 1   0   0   -1   1   1
-1   0   0    1  -1  -1

-1  -2   0   -3  -9   3
 0   1   1    4   2   2
 0   0  -2   -4   4  -8           ditto
 1   0   0   -1   1   1
-1   0   0    1  -1  -1

-1  -1   0   -1  -5   1
 0   1   1    4   2   2     a',c not both null
 0  -1  -2   -6   0  -6
 1   0   0   -1   1   1
-1   0   0    1  -1  -1

-1  -1   0   -1  -5   1     orthogonality implies
 1   0   0   -1   1   1      d_1 =2, d_2 =1
-1   0  -1   -1   1  -5      so a not null
 0   1   1    4   2   2
 0  -1  -1   -4  -2  -2

-1  -2  -1   -5  -7  -1
 1   1   0    1   5  -1     a,c not both null
-1   0   0    1  -1  -1
 0   0   1    2  -2   4
 0   0  -1   -2   2  -4

-1  -2   0   -3  -9   3
 1   1   0    1   5  -1
-1   0  -1   -1   1  -5          ditto
 0   0   1    2  -2   4
 0   0  -1   -2   2  -4
```

****

```
44)       -2  0   0   The vector -1  is also in 2-plane.
           1  1  -1               1
           0 -2   0              -1
           0  0   1               0
           0  0  -1               0
```

This is triangle with midpoint of one edge.

```
          -2   0  -1    0   0  -6     a,c not both null
           1  -1   1   -1  -5   7
           0   0  -1   -2   2  -4
           0   1   0    2   4  -2
           0  -1   0   -2  -4   2

          -1   0  -2   -3   3  -9
           1  -1   1   -1  -5   7          ditto
          -1   0   0    1  -1  -1
           0   1   0    2   4  -2
           0  -1   0   -2  -4   2

           0  -2   0   -4  -8   4
          -1   1   1    5   1   1     orthogonality fails
           0   0  -2   -4   4  -8
           1   0   0   -1   1   1
          -1   0   0    1  -1  -1

           0  -2  -1   -6  -6   0
          -1   1   1    5   1   1     a,c not both null
           0   0  -1   -2   2  -4
           1   0   0   -1   1   1
          -1   0   0    1  -1  -1
```

****

```
45)       -2   0   0       -1  is also in 2-plane.
           1   1   0        1
           0  -2   0       -1
           0   0  -1        0
           0   0  -1        0
           0   0   1        0
```

This is a triangle with midpoint of one edge.

```
          -2   0  -1    0   0  -6
           1   0   1    1  -1   5
           0   0  -1   -2   2  -4    a,c not both null
           0  -1   0   -2  -4   2
           0  -1   0   -2  -4   2
           0   1   0    2   4  -2

          -1   0  -2   -3   3  -9
           1   0   1    1  -1   5
          -1   0   0    1  -1  -1        ditto
           0  -1   0   -2  -4   2
           0  -1   0   -2  -4   2
           0   1   0    2   4  -2
```

```
           0  -2  -1   -6  -6   0
           0   1   1    4   2   2
           0   0  -1   -2   2  -4        ditto
          -1   0   0    1  -1  -1
          -1   0   0    1  -1  -1
           1   0   0   -1   1   1

           0  -2   0   -4  -8   4
           0   1   1    4   2   2
           0   0  -2   -4   4  -8
          -1   0   0    1  -1  -1     This is (Tr 13)
          -1   0   0    1  -1  -1
           1   0   0   -1   1   1
```

****

```
46)       -2   0  -1        equivalent to 11)
           1  -2   1
           0   1   0
           0   0  -1
```

****

```
47)       -2   0  -1   now  -1  is also in 2-plane
           1  -2   0        -1
           0   1   1         0
           0   0  -1         1
```

This is the parallelogram (P4). Subtriangles to consider are

```
          -2   0  -1   0   0  -6
           1  -2   0  -5  -7   5   orthogonality fails
           0   1   1   4   2   2
           0   0  -1  -2   2  -4

          -2   0  -1   0   0  -6
           1  -2  -1  -7  -5   1       ditto
           0   1   0   2   4  -2
           0   0   1   2  -2   4

           0  -2  -1  -6  -6   0
          -2   1   0   4   2  -4
           1   0   1   1  -1   5   a,c not both null
           0   0  -1  -2   2  -4

           0  -2  -1  -6  -6   0
          -2   1  -1   2   4  -8       ditto
           1   0   0  -1   1   1
           0   0   1   2  -2   4

           0  -1  -1  -4  -2  -2
          -2   0  -1   0   0  -6   orthogonality fails
           1   1   0   1   5  -1
           0  -1   1   0  -6   6
```

```
-1  -2   0  -3  -9   3
 0   1  -2  -2   8 -10   a,c not both null
 1   0   1   1  -1   5
-1   0   0   1  -1  -1

-1  -2  -1  -5  -7  -1
 0   1  -1   0   6  -6       ditto
 1   0   0  -1   1   1
-1   0   1   3  -3   3

-1   0  -1  -1   1  -5
 0  -2  -1  -6  -6   0       ditto
 1   1   0   1   5  -1
-1   0   1   3  -3   3

-1  -2   0  -3  -9   3
-1   1  -2  -1   7 -11       ditto
 0   0   1   2  -2   4
 1   0   0  -1   1   1

-1  -2  -1  -5  -7  -1
-1   1   0   3   3  -3       ditto
 0   0   1   2  -2   4
 1   0  -1  -3   3  -3

-1   0  -1  -1   1  -5  orthogonality implies
-1  -2   0  -3  -9   3  d_1=1, d_2 =3 so a'
 0   1   1   4   2   2  is not null
 1   0  -1  -3   3  -3
```

****

47a) Triangle

```
-2     0     -1      0      0    -6
 1    -2      0     -5     -7     5
 0     1     -1      0      6    -6   a,c not both null
 0     0      1      2     -2     4

 0    -2     -1     -6     -6     0
-2     1      0      4      2    -4      ditto
 1     0     -1     -3      3    -3
 0     0      1      2     -2     4

-1    -2      0     -3     -9     3
 0     1     -2     -2      8   -10      ditto
-1     0      1      3     -3     3
-1     0      0     -1      1     1
```

****

47b) Triangle

```
 0    -2      0     -4     -8     4
-2     1     -1      2      4    -8
 1     0     -1     -3      3    -3   a,c not both null
 0     0      1      2     -2     4
```

```
 0    -2     0    -4    -8     4
-1     1    -2    -1     7   -11
-1     0     1     3    -3     3    ditto
 1     0     0    -1     1     1
```

****

48) Triangle

```
-2     0     1     4    -4     2
 1    -2    -1    -7    -5     1   orthogonality fails
 0     1     0     2     4    -2
 0     0    -1    -2     2    -4

 1    -2     0    -5    -7     5
-1     1    -2    -1     7   -11   a,c not both null
 0     0     1     2    -2     4
-1     0     0     1    -1    -1
```

****

49)

```
-2    0    0
 1   -2   -1    equivalent to 3)
 0    1    1
 0    0   -1
```

****

50) Triangle

```
-2     0     1     4    -4     2
 1    -2     0    -5    -7     5    a,c not both null
 0     1    -1     0     6    -6
 0     0    -1    -2     2    -4

 0    -2     1    -2   -10     8
-2     1     0     4     2    -4    a',c not both null
 1     0    -1    -3     3    -3
 0     0    -1    -2     2    -4

 1    -2     0    -5    -7     5
 0     1    -2    -2     8   -10    a,c not both null
-1     0     1     3    -3     3
-1     0     0     1    -1    -1
```

****

51) Triangle

```
 0    -2     0    -4    -8     4
-2     1     1     6     0     0    orthogonality fails
 1     0    -1    -3     3    -3
 0     0    -1    -2     2    -4
```

0	-2	0	-4	-8	4	
1	1	-2	-3	9	-9	a,c not both null
-1	0	1	3	-3	3	
-1	0	0	1	-1	-1	

****

52) Triangle

-2	0	-1	0	0	-6	
1	-2	0	-5	-7	5	a,c not both null
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	
0	0	1	2	-2	4	

-1	0	-2	-3	3	-9	
0	-2	1	-2	-10	8	ditto
0	1	0	2	4	-2	
-1	0	0	1	-1	-1	
1	0	0	-1	1	1	

****

53) Triangle

0	-2	0	-4	-8	4	
-1	1	-2	-1	7	-11	
0	0	1	2	-2	4	a,c not both null
1	0	0	-1	1	1	
-1	0	0	1	-1	-1	

****

54) Triangle

0	-2	0	-4	-8	4	
0	1	-2	-2	8	-10	
-1	0	1	-3	3	-3	a,c not both null
-1	0	0	1	-1	-1	
1	0	0	-1	1	1	

0	-2	0	-4	-8	4	
-2	1	0	4	2	-4	
1	0	-1	-3	3	-3	a', c not both null
0	0	-1	-2	2	-4	
0	0	1	2	-2	4	

****

55) Triangle

-2	0	1	4	-4	2	
1	-2	0	-5	-7	5	
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	a,c not both null
0	0	-1	-2	2	-4	

1	0	-2	-5	5	-7	
0	-2	1	-2	-10	8	
0	1	0	2	4	-2	ditto
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

56) Triangle

0	-2	0	-4	-8	4	
-2	1	1	6	0	0	
1	0	0	-1	1	1	orthogonality fails
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
1	1	-2	-3	9	-9	
0	0	1	2	-2	4	a,c not both null
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

57) Triangle

0	-2	0	-4	-8	4	
-2	1	0	4	2	-4	
1	0	1	1	-1	5	a', c not both null
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
0	1	-2	-2	8	-10	
1	0	1	1	-1	5	ditto
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

58) Triangle

0	-2	0	-4	-8	4	
0	1	-2	-2	8	-10	
0	0	1	2	-2	4	a,c not both null
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	
1	0	0	-1	1	1	

****

59)

-2	0	-1	
1	0	1	equivalent to 2)
0	-2	-1	
0	1	0	

****

60) Triangle

-2	0	-1	0	0	-6	
1	0	-1	-3	3	-3	a,c not both null
0	-2	1	-2	-10	8	
0	1	0	2	4	-2	

-1	0	-2	-3	3	-9	
-1	0	1	3	-3	3	
1	-2	0	-5	-7	5	ditto
0	1	0	2	4	-2	

****

61) Triangle

-2	0	-1	0	0	-6	
1	0	-1	-3	3	-3	orthogonality implies
0	-2	0	-4	-8	4	d_2 =1, now
0	1	1	4	2	2	a' is not null

-1	0	-2	-3	3	-9	
-1	0	1	3	-3	3	a,c not both null
0	-2	0	-4	-8	4	
1	1	0	1	5	-1	

****

62)

-2	0	-1	
1	0	1	equivalent to 3)
0	-2	0	
0	1	-1	

****

63)

-2	0	-1	Now	1	also lies in face but the parallelogram
1	0	0		-1	
0	-2	1		-1	
0	1	-1		0	

formed is not a face. Subtriangles are

-2	-1	1	2	-8	4	
1	0	-1	-3	3	-3	a, c not both null
0	1	-1	0	6	-6	
0	-1	0	-2	-4	2	

-2	0	1	4	-4	2	
1	0	-1	-3	3	-3	ditto
0	-2	-1	-6	-6	0	
0	1	0	2	4	-2	

1	-2	0	-5	-7	5	
-1	1	0	3	3	-3	ditto
-1	0	-2	-3	3	-9	
0	0	1	2	-2	4	

1	-2	-1	-7	-5	1	
-1	1	0	3	3	-3	ditto
-1	0	1	3	-3	3	
0	0	-1	-2	2	-4	

1	0	-1	-3	3	-3	
-1	0	0	1	-1	-1	ditto
-1	-2	1	-1	-11	7	
0	1	-1	0	6	-6	

****

64) Triangle

-2	0	1	4	-4	2	
1	0	-1	-3	3	-3	
0	-2	0	-4	-8	4	a,c not both null
0	1	-1	0	6	-6	

1	0	-2	-5	5	-7	
-1	0	1	3	-3	3	ditto
0	-2	0	-4	-8	4	
-1	1	0	3	3	-3	

****

65) Triangle

-1	-2	0	-3	-9	3	
0	1	0	2	4	-2	a,c not both null
-1	0	-2	-3	3	-9	
0	0	1	2	-2	4	
1	0	0	-1	1	1	

****

66) Triangle

-2	0	-1	0	0	-6	
1	0	-1	-3	3	-3	a,c not both null
0	-2	0	-4	-8	4	
0	1	0	2	4	-2	
0	0	1	2	-2	4	

0	-1	-2	-6	0	-6	
0	-1	1	0	-6	6	
-2	0	0	2	-2	-2	a', c not both null
1	0	0	-1	1	1	
0	1	0	2	4	-2	

-1	-2	0	-3	-9	3	
-1	1	0	3	3	-3	a,c not both null
0	0	-2	-4	4	-8	
0	0	1	2	-2	4	
1	0	0	-1	1	1	

****

67) Triangle

-1	-2	0	-3	-9	3	
0	1	0	2	4	-2	
0	0	-2	-4	4	-8	a,c not both null
-1	0	1	3	-3	3	
1	0	0	-1	1	1	

****

68) Triangle

0	-2	0	-4	-8	4	
-1	1	0	3	3	-3	
0	0	-2	-4	4	-8	a,c not both null
-1	0	1	3	-3	3	
1	0	0	-1	1	1	

****

69) Triangle

-1	-2	0	-3	-9	3	
0	1	0	2	4	-2	a,c not both null
1	0	-2	-5	5	-7	
0	0	1	2	-2	4	
-1	0	0	1	-1	-1	

****

70)

-2	0	-1	
1	0	1	equivalent to 14)
0	-2	0	
0	1	0	
0	0	-1	

****

71) Triangle

-1	-2	0	-3	-9	3	
0	1	0	2	4	-2	
0	0	-2	-4	4	-8	a,c not both null
1	0	1	1	-1	5	
-1	0	0	1	-1	-1	

****

72) Triangle

-2	0	1	4	-4	2	
1	0	-1	-3	3	-3	
0	-2	0	-4	-8	4	a,c not both null
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	

0	1	-2	-2	8	-10	
0	-1	1	0	-6	6	
-2	0	0	2	-2	-2	ditto
1	0	0	-1	1	1	
0	-1	0	-2	-4	2	

1	-2	0	-5	-7	5	
-1	1	0	3	3	-3	
0	0	-2	-4	4	-8	ditto
0	0	1	2	-2	4	
-1	0	0	1	-1	-1	

****

73) Triangle

1	-2	0	-5	-7	5	
0	1	0	2	4	-2	a,c not both null
0	0	-2	-4	4	-8	
-1	0	1	3	-3	3	
-1	0	0	1	-1	-1	

****

74) Triangle

0	-2	0	-4	-8	4	
1	1	0	1	5	-1	
0	0	-2	-4	4	-8	ditto
-1	0	1	3	-3	3	
-1	0	0	1	-1	-1	

****

75) Triangle

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	-2	-1	-6	-6	0	a,c not both null
0	1	0	2	4	-2	
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	
-1	0	-2	-3	3	-9	ditto
0	0	1	2	-2	4	
1	0	0	-1	1	1	
-1	0	0	1	-1	-1	

****

76) Triangle

-2	0	0	2	-2	-2
1	0	0	-1	1	1

0	-2	0	-4	-8	4	a,c not both null
0	1	-1	0	6	-6	
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	
0	0	-2	-4	4	-8	ditto
-1	0	1	3	-3	3	
1	0	0	-1	1	1	
-1	0	0	1	-1	-1	

****

77) Triangle

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	-2	0	-4	-8	4	this is (Tr4)
0	1	1	4	2	2	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	0	-2	-4	4	-8	
0	0	1	2	-2	4	
0	-2	0	-4	-8	4	a,c not both null
1	1	0	1	5	-1	
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

78) Triangle

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	-2	1	-2	-10	8	a,c not both null
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	0	-2	-4	4	-8	
0	0	1	2	-2	4	
1	-2	0	-5	-7	5	
0	1	0	2	4	-2	ditto
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

79) Triangle

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	-2	0	-4	-8	4	
0	1	0	2	4	-2	a,c not both null
0	0	-1	-2	2	-4	

0	0	-1	-2	2	-4	
0	0	1	2	-2	4	

0	0	-2	-4	4	-8	
0	0	1	2	-2	4	
-2	0	0	2	-2	-2	
1	0	0	-1	1	1	ditto
0	-1	0	-2	-4	2	
0	-1	0	-2	-4	2	
0	1	0	2	4	-2	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	
0	0	-2	-4	4	-8	
0	0	1	2	-2	4	ditto
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	
1	0	0	-1	1	1	

****

Now consider 2-planes including a type III, a type II and a type I

80)	-2	-1	0	equivalent to 20)
	1	1	0	
	0	-1	0	
	0	0	-1	

****

81)	-2	1	0	2-plane also contains	-1
	1	-1	0		0
	0	-1	0		-1
	0	0	-1		1

and gives a parallelogram, but not a face.
Subtriangles to consider are

-2	1	-1	2	4	-8	
1	-1	0	-3	-3	3	orthogonality fails
0	-1	-1	-4	-2	-2	
0	0	1	2	-2	4	

-2	1	0	4	2	-4	
1	-1	0	-3	-3	3	a',c not both null
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	

1	-2	0	-5	-7	5	
-1	1	0	3	3	-3	
-1	0	0	1	-1	-1	a,c not both null
0	0	-1	-2	2	-4	

1	-2	-1	-7	-5	1	
-1	1	0	3	3	-3	
-1	0	-1	-1	1	-5	ditto
0	0	1	2	-2	4	

1	0	-1	-3	3	-3	
-1	0	0	1	-1	-1	
-1	0	-1	-1	1	-5	ditto
0	-1	1	0	-6	6	

-1	-2	1	-1	-11	7	
0	1	-1	0	6	-6	ditto
-1	0	-1	-1	1	-5	
1	0	0	-1	1	1	

-1	-2	0	-3	-9	3	
0	1	0	2	4	-2	ditto
-1	0	0	1	-1	-1	
1	0	-1	-3	3	-3	

-1	1	0	3	3	-3	
0	-1	0	-2	-4	2	
-1	-1	0	-1	-5	1	ditto
1	0	-1	-3	3	-3	

****

82) Triangle

-2	-1	0	0	-6	0	
1	-1	0	-3	-3	3	a,c not both null
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	

-1	-2	0	-3	-9	3	
-1	1	0	3	3	-3	ditto
1	0	0	-1	1	1	
0	0	-1	-2	2	-4	

****

83)

-2	-1	0	equivalent to 81)
1	0	0	
0	-1	0	
0	1	-1	

****

84)

-2	-1	0	equivalent to 20)
1	0	0	
0	-1	-1	
0	1	0	

****

85)

-2	0	0	
1	-1	0	equivalent to 16)
0	-1	-1	
0	1	0	

****

86) Triangle

0	-2	0	-4	-8	4	
-1	1	0	3	3	-3	a,c not both null
-1	0	0	1	-1	-1	
1	0	-1	-3	3	-3	

****

87)

-2	0	0	
1	1	0	equivalent to 16)
0	-1	-1	
0	-1	0	

****

88)

-2	1	0	
1	0	0	equivalent to 20)
0	-1	-1	
0	-1	0	

****

89) Triangle

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	1	-1	0	6	-6	a,c not both null
0	-1	0	-2	-4	2	
0	-1	0	-2	-4	2	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	ditto
1	0	-1	-3	3	-3	
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

90)

-2	0	0	0	is also in 2-plane.
1	0	0	0	
0	1	0	-1	
0	-1	-1	-1	
0	-1	0	1	

We have triangle with a midpoint of one edge.

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	1	0	2	4	-2	a',c not both null
0	-1	-1	-4	-2	-2	
0	-1	0	-2	-4	2	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	a,c not both null
1	0	0	-1	1	1	
-1	0	-1	-1	1	-5	
-1	0	0	1	-1	-1	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	
1	0	-1	-3	3	-3	ditto
-1	0	-1	-1	1	-5	
-1	0	1	3	-3	3	

****

90a) Triangle

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	1	0	2	4	-2	a,c not both null
0	-1	0	-2	-4	2	
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	ditto
1	0	0	-1	1	1	
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	

****

We do not have to consider the situation with a type III and two type I
since affine combinations of the type I
-1 0 give -2 1
0 -1 1 -2
and so will have already been considered before.

Now consider 2-planes including a type III and two type II.
Note that configurations involving -2 -1 need not be considered
1 1
0 -1
. .
as then 0 is also in the 2-plane so the example will have already occurred.
1
-2
.

For example

91) -2 -1 0 has 0 in the 2-plane so is equivalent to 12).
1 1 1 1
0 -1 -1 -2
0 0 -1 0

****

```
92)     -2    1   0     equivalent to 38)
         1   -1  -1
         0   -1  -1
         0    0   1
```

****

```
93)     -2    1  -1      0   is in the 2-plane.
         1   -1  -1      1
         0   -1   0     -1
         0    0   1     -1
```

This is a parallelogram, but not a face. Nor can the subtriangles be faces.

****

```
94)     -2   1   1         Triangle, but not face
         1  -1  -1
         0  -1   0
         0   0  -1
```

****

```
95)     -2   1  -1
         1  -1   0         equivalent to 81)
         0  -1  -1
         0   0   1
```

****

```
96)     -2   1   1
         1  -1   0
         0  -1  -1         equivalent to 26)
         0   0  -1
```

****

```
97)     -2   1  -1
         1  -1   1         equivalent to 38)
         0  -1   0
         0   0  -1
```

****

```
98)     -2   1   0
         1  -1   1         equivalent to 93)
         0  -1  -1
         0   0  -1
```

****

```
99)     -2   1   0
         1  -1  -1         triangle, not face
         0  -1   1
         0   0  -1
```

****

100)	-2	1	1	equivalent to 63)
	1	-1	0	
	0	-1	0	
	0	0	-1	
	0	0	-1	

****

101)	-2	1	1	
	1	-1	0	
	0	-1	0	triangle, not face
	0	0	-1	
	0	0	-1	

****

102)	-2	1	0	
	1	-1	1	
	0	-1	0	triangle, not face
	0	0	-1	
	0	0	-1	

****

103)	-2	1	0	
	1	-1	0	
	0	-1	1	triangle , not face
	0	0	-1	
	0	0	-1	

****

104)	-2	1	-1	now 0	is in the 2-plane.
	1	-1	0	0	
	0	-1	0	-1	
	0	0	-1	1	
	0	0	1	-1	

If the first two or last two are present it is not a face , so no subtriangle is a face.

****

105)	-2	1	0	triangle, not face
	1	-1	-1	
	0	-1	0	
	0	0	-1	
	0	0	1	

****

106)	-2	1	0	equivalent to 104)
	1	-1	0	
	0	-1	-1	
	0	0	-1	
	0	0	1	

****

107) Triangle

-2	-1	-1	-2	-4	-4	
1	-1	-1	-5	-1	-1	orthogonality implies
0	1	0	2	4	-2	(d_1,d_2)=(2,1), which
0	0	1	2	-2	4	contradicts nullity of c

-1	-2	-1	-5	-7	-1	
-1	1	-1	1	5	-7	
1	0	0	-1	1	1	a,c not both null
0	0	1	2	-2	4	

-1	-2	-1	-5	-7	-1	
-1	1	-1	1	5	-7	a,c not both null
0	0	1	2	-2	4	
1	0	0	-1	1	1	

****

108)

-2	-1	-1	equivalent to 47)
1	-1	-1	
0	1	-1	
0	0	1	

****

109) Triangle

-1	-2	0	-3	-9	3	
-1	1	-1	1	5	-7	
1	0	-1	-3	3	-3	a,c not both null
0	0	1	2	-2	4	

0	-2	-1	-6	-6	0	
-1	1	-1	1	5	-7	ditto
-1	0	1	3	-3	3	
1	0	0	-1	1	1	

****

110)

-2	-1	-1	equivalent to 40)
1	-1	1	
0	1	0	
0	0	-1	

****

111)

-2	-1	-1	equivalent to 24)
1	-1	0	
0	1	1	
0	0	-1	

****

112)

-2	-1	0	equivalent to 110)
1	-1	-1	
0	1	1	
0	0	-1	

****

113)

-2	-1	1	equivalent to 93)
1	-1	-1	
0	1	0	
0	0	-1	

****

114)

-2	-1	0	equivalent to 113)
1	-1	1	
0	1	-1	
0	0	-1	

****

115) Triangle

-2	-1	1	2	-8	4	
1	-1	0	-3	-3	3	a,c not both null
0	1	-1	0	6	-6	
0	0	-1	-2	2	-4	

-1	-2	1	-1	-11	7	
-1	1	0	3	3	-3	ditto
1	0	-1	-3	3	-3	
0	0	-1	-2	2	-4	

1	-2	-1	-7	-5	1	
0	1	-1	0	6	-6	a,a' not both null
-1	0	1	3	-3	3	
-1	0	0	1	-1	-1	

****

116) Triangle

-1	-2	-1	-5	-7	-1	
-1	1	0	3	3	-3	
1	0	0	-1	1	1	a,c not both null
0	0	-1	-2	2	-4	
0	0	1	2	-2	4	

-1	-2	-1	-5	-7	-1	
0	1	-1	0	6	-6	
0	0	1	2	-2	4	ditto
-1	0	0	1	-1	-1	
1	0	0	-1	1	1	

****

117) Triangle

-1	-2	0	-3	-9	3	
-1	1	-1	1	5	-7	
1	0	0	-1	1	1	a,c not both null
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

0	-2	-1	-6	-6	0	
-1	1	-1	1	5	-7	
0	0	1	2	-2	4	ditto
1	0	0	-1	1	1	
-1	0	0	1	-1	-1	

****

118) Triangle

-2	-1	0	0	-6	0	
1	-1	0	-3	-3	3	
0	1	-1	0	6	-6	a,c not both null
0	0	1	2	-2	-4	
0	0	-1	-2	2	-4	

-1	-2	0	-3	-9	3	
-1	1	0	3	3	-3	
1	0	-1	-3	3	-3	ditto
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

0	-2	-1	-6	-6	0	
0	1	-1	0	6	-6	ditto
-1	0	1	3	-3	3	
1	0	0	-1	1	1	
-1	0	0	1	-1	-1	

****

119) Triangle

-2	-1	0	0	-6	0	
1	-1	0	-3	-3	3	orthogonality implies
0	1	1	4	2	2	d_2 =1, so a not null
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

-1	-2	0	-3	-9	3	
-1	1	0	3	3	-3	
1	0	1	1	-1	5	a,c not both null
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	-2	-1	-6	-6	0	
0	1	-1	0	6	-6	
1	0	1	1	-1	5	ditto
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

120) Triangle

-2	-1	0	0	-6	0	
1	-1	1	-1	-5	7	
0	1	0	2	4	-2	a,c not both null
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

-1	-2	0	-3	-9	3	
-1	1	1	5	1	1	
1	0	0	-1	1	1	orthogonality fails
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	-2	-1	-6	-6	0	
1	1	-1	-1	7	-5	
0	0	1	2	-2	4	a,c not both null
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

121) Triangle

-2	-1	1	2	-8	4	
1	-1	0	-3	-3	3	
0	1	0	2	4	-2	a,c not both null
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

-1	-2	1	-1	-11	7	
-1	1	0	3	3	-3	ditto
1	0	0	-1	1	1	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

1	-2	-1	-7	-5	1	
0	1	-1	0	6	-6	
0	0	1	2	-2	4	orthogonality fails
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

122)

-2	-1	0	equivalent to 33)
1	0	-1	
0	1	1	
0	-1	-1	

****

123)

-2	-1	0	
1	0	-1	equivalent to 33)
0	1	-1	
0	-1	1	

****

```
124)    -2  -1   0         now  -1 is in 2-plane
         1   0  -1               0
         0   1   1               0
         0  -1   0               1
         0   0  -1              -1

         This is parallelogram (P5).

        -2  -1   0     0  -6   0
         1   0  -1    -3   3  -3
         0   1   1     4   2   2   orthogonality fails
         0  -1   0    -2  -4   2
         0   0  -1    -2   2  -4

        -2  -1  -1    -2  -4  -4
         1   0   0    -1   1   1
         0   1   0     2   4  -2   a,c not both null
         0  -1   1     0  -6   6
         0   0  -1    -2   2  -4

        -2   0  -1     0   0  -6
         1  -1   0    -3  -3   3   orthogonality fails
         0   1   0     2   4  -2
         0   0   1     2  -2   4
         0  -1  -1    -4  -2  -2

        -1  -2   0    -3  -9   3
         0   1  -1     0   6  -6   a,c not both null
         1   0   1     1  -1   5
        -1   0   0     1  -1  -1
         0   0  -1    -2   2  -4

        -1  -2  -1    -5  -7  -1
         0   1   0     2   4  -2
         1   0   0    -1   1   1          ditto
        -1   0   1     3  -3   3
         0   0  -1    -2   2  -4

        -1   0  -1    -1   1  -5
         0  -1   0    -2  -4   2
         1   1   0     1   5  -1   orthogonality implies {d_1,d_3}
        -1   0   1     3  -3   3    = {1,2} so a not null
         0  -1  -1    -4  -2  -2

         0  -2  -1    -6  -6   0
        -1   1   0     3   3  -3
         1   0   0    -1   1   1   orthogonality fails
         0   0   1     2  -2   4
        -1   0  -1    -1   1  -5

         0  -2  -1    -6  -6   0
        -1   1   0     3   3  -3   orthogonality fails
         1   0   1     1  -1   5
         0   0  -1    -2   2  -4
        -1   0   0     1  -1  -1
```

```
         0  -1  -1     -4  -2  -2   orthogonality implies d_2=1, d_3=2
        -1   0   0      1  -1  -1      so a' is not null
         1   0   1      1  -1   5
         0   1  -1      0   6  -6
        -1  -1   0     -1  -5   1

        -1  -2  -1     -5  -7  -1
         0   1   0      2   4  -2
         0   0   1      2  -2   4   a,c not both null
         1   0  -1     -3   3  -3
        -1   0   0      1  -1  -1

        -1  -2   0     -3  -9   3
         0   1  -1      0   6  -6
         0   0   1      2  -2   4
         1   0   0     -1   1   1           ditto
        -1   0  -1     -1   1  -5

        -1  -1   0     -1  -5   1  orthogonality implies {d_1,d_5}
         0   0  -1     -2   2  -4     = {1,2}, so a not null
         0   1   1      4   2   2
         1  -1   0     -3  -3   3
        -1   0  -1     -1   1  -5

                                  ***************

125)    -2  -1   0      now    -1  is in 2-plane
         1   0  -1              0
         0  -1  -1              0
         0   1   0             -1
         0   0   1              1

        This is parallelogram (P6)

        -2  -1   0    0  -6   0
         1   0  -1   -3   3  -3  orthogonality fails
         0  -1  -1   -4  -2  -2
         0   1   0    2   4  -2
         0   0   1    2  -2   4

        -2  -1  -1   -2  -4  -4
         1   0   0   -1   1   1
         0  -1   0   -2  -4   2  a,c not both null
         0   1  -1    0   6  -6
         0   0   1    2  -2   4

        -2   0  -1    0   0  -6
         1  -1   0   -3  -3   3  orthogonality fails
         0  -1   0   -2  -4   2
         0   0  -1   -2   2  -4
         0   1   1    4   2   2

        -1  -2   0   -3  -9   3
         0   1  -1    0   6  -6
        -1   0  -1   -1   1  -5   a,c not both null
         1   0   0   -1   1   1
         0   0   1    2  -2   4
```

```
-1  -2  -1   -5  -7  -1
 0   1   0    2   4  -2
-1   0   0    1  -1  -1   a,c not both null
 1   0  -1   -3   3  -3
 0   0   1    2  -2   4

-1   0  -1   -1   1  -5   orthogonality implies d_1=1
 0  -1   0   -2  -4   2    d_3 = 2 and now a'
-1  -1   0   -1  -5   1    is not null
 1   0  -1   -3   3  -3
 0   1   1    4   2   2

 0  -2  -1   -6  -6   0
-1   1   0    3   3  -3
-1   0  -1   -1   1  -5   orthogonality fails
 0   0   1    2  -2   4
 1   0   0   -1   1   1

 0  -2  -1   -6  -6   0
-1   1   0    3   3  -3
-1   0   0    1  -1  -1   orthogonality fails
 0   0  -1   -2   2  -4
 1   0   1    1  -1   5

 0  -1  -1   -4  -2  -2
-1   0   0    1  -1  -1   orthogonality implies
-1  -1   0   -1  -5   1    d_3 =2, d_2 =1 so a is not null
 0   1  -1    0   6  -6
 1   0   1    1  -1   5

-1  -2  -1   -5  -7  -1
 0   1   0    2   4  -2
 0   0  -1   -2   2  -4   a,c not both null
-1   0   1    3  -3   3
 1   0   0   -1   1   1

-1  -2   0   -3  -9   3
 0   1  -1    0   6  -6        ditto
 0   0  -1   -2   2  -4
-1   0   0    1  -1  -1
 1   0   0    1  -1   5

-1  -1   0   -1  -5   1   orthogonality  implies
 0   0  -1   -2   2  -4     d_1 =1, d_5 = 2
 0  -1  -1   -4  -2  -2     and a' is not null
-1   1   0    3   3  -3
 1   0   1    1  -1   5
```

****

```
126)    Triangle

        -2  -1   0    0  -6   0
         1   0  -1   -3   3  -3   a,c not both null
         0  -1   1    0  -6   6
         0   1   0    2   4  -2
         0   0  -1   -2   2  -4
```

```
-1  -2   0    -3  -9   3
 0   1  -1     0   6  -6         ditto
-1   0   1     3  -3   3
 1   0   0    -1   1   1
 0   0  -1    -2   2  -4

 0  -2  -1    -6  -6   0
-1   1   0     3   3  -3
 1   0  -1    -3   3  -3  orthogonality fails
 0   0   1     2  -2   4
-1   0   0     1  -1  -1
```

****

127) Triangle

```
-2  -1   0     0  -6   0
 1   0  -1    -3   3  -3
 0   1   0     2   4   2    a,c not both null
 0  -1   0    -2  -4   2
 0   0   1     2  -2   4
 0   0  -1    -2   2  -4

-1  -2   0    -3  -9   3
 0   1  -1     0   6  -6
 1   0   0    -1   1   1         ditto
-1   0   0     1  -1  -1
 0   0   1     2  -2   4
 0   0  -1    -2   2  -4

 0  -2  -1    -6  -6   0
-1   1   0     3   3  -3
 0   0   1     2  -2   4
 0   0  -1    -2   2  -4   orthogonality fails
 1   0   0    -1   1   1
-1   0   0     1  -1  -1
```

****

128)

```
-2   1   0          equivalent to 34)
 1   0   1
 0  -1  -1
 0  -1  -1
```

****

129)

```
-2   1   0          now  -1 is in 2-plane
 1   0   1                0
 0  -1  -1                0
 0   0  -1                1
 0  -1   1               -1
```

This is parallelogram (P7).

-2	1	0	4	2	-4	
1	0	1	1	-1	5	
0	-1	-1	-4	-2	-2	orthogonality fails
0	0	-1	-2	2	-4	
0	-1	0	-2	-4	2	

-2	-1	0	0	-6	0	
1	0	1	1	-1	5	
0	0	-1	-2	2	-4	a,c not both null
0	1	-1	0	6	-6	
0	-1	0	-2	-4	2	

1	-2	0	-5	-7	5	
0	1	1	4	2	2	
-1	0	-1	-1	1	-5	orthgonality fails
0	0	-1	-2	2	-4	
-1	0	0	1	-1	-1	

1	-2	-1	-7	-5	1	
0	1	0	2	4	-2	
-1	0	0	1	-1	-1	orthogonality fails
0	0	1	2	-2	4	
-1	0	-1	-1	1	-5	

1	0	-1	-3	3	-3	
0	1	0	2	4	-2	
-1	-1	0	-1	-5	1	a,c not both null
0	-1	1	0	-6	6	
-1	0	-1	-1	1	-5	

0	-2	1	-2	-10	8	
1	1	0	1	5	-1	
-1	0	-1	-1	1	-5	ditto
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	

0	-2	-1	-6	-6	0	
1	1	0	1	5	-1	
-1	0	0	1	-1	-1	ditto
-1	0	1	3	-3	3	
0	0	-1	-2	2	-4	

0	1	-1	0	6	-6	
1	0	0	-1	1	1	
-1	-1	0	-1	-5	1	a,a' not both null
-1	0	1	3	-3	3	
0	-1	-1	-4	-2	-2	

-1	-2	1	-1	-11	7	
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	a,c not both null
1	0	0	-1	1	1	
-1	0	0	-1	1	-5	

-1	-2	0	-3	-9	3	
0	1	1	4	2	2	
0	0	-1	-2	2	-4	orthogonality fails
1	0	-1	-3	3	-3	
-1	0	0	1	-1	-1	

-1	1	0	3	3	-3	
0	0	1	2	-2	4	
0	-1	-1	-4	-2	-2	orthogonality fails
0	0	-1	-3	3	-3	
-1	-1	0	-1	-5	1	

****

130) Triangle

-2	1	0	4	2	-4	
1	0	1	1	-1	5	a', c not both null
0	-1	0	-2	-4	2	
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

1	-2	0	-5	-7	5	
0	1	1	4	2	2	
-1	0	0	1	-1	-1	orthogonality fails
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	-2	1	-2	-10	8	
1	1	0	1	5	-1	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	a, c not both null
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

131)

-2	-1	0	equivalent to 47)
1	0	1	
0	-1	-1	
0	1	-1	

****

132) Triangle

-2	-1	0	0	-6	0	
1	0	1	1	-1	5	
0	-1	-1	-4	-2	-2	orthogonality fails
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	

-1	-2	0	-3	-9	3	
0	1	1	4	2	2	
-1	0	-1	-1	1	-5	orthogonality fails
1	0	0	-1	1	1	
0	0	-1	-2	2	-4	

0	-2	-1	-6	-6	0	
1	1	0	1	5	-1	
-1	0	-1	-1	1	-5	a,c not both null
0	0	1	2	-2	4	
-1	0	0	1	-1	-1	

****

133)

-2	-1	0	equivalent to 129)
1	0	1	
0	1	-1	
0	0	-1	
0	-1	0	

****

134) Triangle

-2	-1	0	0	-6	0	
1	0	1	1	-1	5	
0	1	0	2	4	-2	a,c not both null
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

-1	-2	0	-3	-9	3	
0	1	1	4	2	2	
1	0	0	-1	1	1	orthogonality fails
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	-2	1	-6	-6	0	
1	1	0	1	5	-1	
0	0	1	2	-2	4	a, c not both null
0	0	-1	-2	2	-4	
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

****

135) Triangle

-2	1	0	4	2	-4	
1	0	-1	-3	3	-3	a,a' not both null
0	-1	-1	-4	-2	-2	
0	-1	1	0	-6	6	

1	-2	0	-5	-7	5	
0	1	-1	0	6	-6	a,c not both null
-1	0	-1	-1	1	-5	
-1	0	1	3	-3	3	

0	-2	1	-2	-10	8	
-1	1	0	3	3	-3	ditto
-1	0	-1	-1	1	-5	
1	0	-1	-3	3	-3	

****

136) Triangle

-2	1	0	4	2	-4	
1	0	-1	-3	3	-3	
0	-1	-1	-4	-2	-2	orthogonality fails
0	-1	0	-2	-4	2	
0	0	1	2	-2	4	

1	-2	0	-5	-7	5	
0	1	-1	0	6	-6	
-1	0	-1	-1	1	-5	a,c not both null
-1	0	0	1	-1	-1	
0	0	1	2	-2	4	

0	-2	1	-2	-10	8	
-1	1	0	3	3	-3	
-1	0	-1	-1	1	-5	ditto
0	0	-1	-2	2	-4	
1	0	0	-1	1	1	

****

137) Triangle

-2	1	0	4	2	-4	
1	0	-1	-3	3	-3	
0	-1	1	0	-6	6	a', c not both null
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	

1	-2	0	-5	-7	5	
0	1	-1	0	6	-6	
-1	0	1	3	-3	3	a,c not both null
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	

0	-2	1	-2	-10	8	
-1	1	0	3	3	-3	
1	0	-1	-3	3	-3	ditto
0	0	-1	-2	2	-4	
-1	0	0	1	-1	-1	

****

138) Triangle

-2	1	0	4	2	-4	
1	0	-1	-3	3	-3	
0	-1	0	-2	-4	2	a',c not both null
0	-1	0	-2	-4	2	
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

1	-2	0	-5	-7	5	
0	1	-1	0	6	-6	
-1	0	0	1	-1	-1	a,c not both null
-1	0	0	1	-1	-1	
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

0	-2	1	-2	-10	8	
-1	1	0	3	3	-3	
0	0	-1	-2	2	-4	ditto
0	0	-1	-2	2	-4	
1	0	0	-1	1	1	
-1	0	0	1	-1	-1	

****

139)

-2	-1	-1	equivalent to 33)
1	0	0	
0	1	-1	
0	-1	1	

****

140) Triangle

-2	-1	-1	-2	-4	-4	
1	0	0	-1	1	1	orthogonality and nullity
0	1	1	4	2	2	eqns have no integral solution
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	

-1	-2	-1	-5	-7	-1	
0	1	0	2	4	-2	
1	0	1	1	-1	5	a,c not both null
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	

-1	-2	-1	-5	-7	-1	
0	1	0	2	4	-2	
1	0	1	1	-1	5	ditto
0	0	-1	-2	2	-4	
-1	0	0	1	-1	-1	

****

141) Triangle

-2	-1	-1	-2	-4	-4	
1	0	0	-1	1	1	orthogonality and nullity relations
0	1	0	2	4	-2	have no integer solution
0	-1	-1	-4	-2	-2	
0	0	1	2	-2	4	

-1	-2	-1	-5	-7	-1	
0	1	0	2	4	-2	
1	0	0	-1	1	1	a,c not both null
-1	0	-1	-1	1	-5	
0	0	1	2	-2	4	

```
        -1  -2  -1    -5  -7  -1
         0   1   0     2   4  -2
         0   0   1     2  -2   4              ditto
        -1   0  -1    -1   1  -5
         1   0   0    -1   1   1
```

****

```
142)    -2  -1  -1        equivalent to 125)
         1   0   0
         0   1  -1
         0  -1   0
         0   0   1
```

****

```
143)     Triangle

        -2  -1  -1    -2  -4  -4
         1   0   0    -1   1   1
         0   0   1     2  -2   4    a,c not both null
         0   0  -1    -2   2  -4
         0   1   0     2   4  -2
         0  -1   0    -2  -4   2

        -1  -2  -1    -5  -7  -1
         0   1   0     2   4  -2
         0   0   1     2  -2   4           ditto
         0   0  -1    -2   2  -4
         1   0   0    -1   1   1
        -1   0   0     1  -1  -1

        -1  -2  -1    -5  -7  -1
         0   1   0     2   4  -2
         1   0   0    -1   1   1           ditto
        -1   0   0     1  -1  -1
         0   0   1     2  -2   4
         0   0  -1    -2   2  -4
```

****

```
144)     Triangle

        -2   1   1     6   0   0
         1   0   0    -1   1   1
         0  -1  -1    -4  -2  -2    orthogonality fails
         0  -1   0    -2  -4   2
         0   0  -1    -2   2  -4

         1  -2   1    -3  -9   9
         0   1   0     2   4  -2
        -1   0  -1    -1   1  -5    a,c not both null
        -1   0   0     1  -1  -1
         0   0  -1    -2   2  -4
```

1	-2	1	-3	-9	9	
0	1	0	2	4	-2	
-1	0	-1	-1	1	-5	a,c not both null
0	0	-1	-2	2	-4	
-1	0	0	1	-1	-1	

****

145) Triangle

-2	1	1	6	0	0	
1	0	0	-1	1	1	
0	-1	0	-2	-4	2	orthogonality fails
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

1	-2	1	-3	-9	9	
0	1	0	2	4	-2	
-1	0	0	1	-1	-1	a,c not both null
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

****

146)

-2	0	0	equivalent to 33)
1	-1	-1	
0	-1	1	
0	1	-1	

****

147) Triangle

-2	0	0	2	-2	-2	
1	-1	-1	-5	-1	-1	orthogonality fails
0	-1	-1	-4	-2	-2	as d_1 is not 1
0	1	0	2	4	-2	
0	0	1	2	-2	4	

0	-2	0	-4	-8	4	
-1	1	-1	1	5	-7	
-1	0	-1	-1	1	-5	a,c not both null
1	0	0	-1	1	1	
0	0	1	2	-2	4	

0	-2	0	-4	-8	4	
-1	1	-1	1	5	-7	
-1	0	-1	-1	1	-5	ditto
0	0	1	2	-2	4	
1	0	0	-1	1	1	

****

148) Triangle

-2	0	0	2	-2	-2	
1	-1	-1	-5	-1	-1	orthogonality fails
0	-1	0	-2	-4	2	as d_1 is not 1
0	1	1	4	2	2	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
-1	1	-1	1	5	-7	
-1	0	0	1	-1	-1	a,c not both null
1	0	1	1	-1	5	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
-1	1	-1	1	5	-7	
0	0	-1	-2	2	-4	a,c not both null
1	0	1	1	-1	5	
-1	0	0	1	- 1	-1	

****

149) Triangle

-2	0	0	2	-2	-2	
1	-1	-1	-5	-1	-1	orthogonality fails
0	-1	1	0	-6	6	as d_1 is not 1
0	1	0	2	4	-2	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
-1	1	-1	1	5	-7	
-1	0	1	3	-3	3	a,c not both null
1	0	0	-1	1	1	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
-1	1	-1	1	5	-7	
1	0	-1	-3	3	-3	ditto
0	0	1	2	-2	4	
-1	0	0	1	-1	-1	

****

150) Triangle

-2	0	0	2	-2	-2	
1	-1	-1	-5	-1	-1	
0	-1	0	-2	-4	2	orthogonality fails,
0	1	0	2	4	-2	since d_1 is not 1
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
-1	1	-1	1	5	-7	
-1	0	0	1	-1	-1	a,c not both null
1	0	0	-1	1	1	
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

```
        0  -2   0    -4  -8   4
       -1   1  -1     1   5  -7
        0   0  -1    -2   2  -4        ditto
        0   0   1     2  -2   4
        1   0   0    -1   1   1
       -1   0   0     1  -1  -1
```

****

We next study configurations involving patterns

```
151)           -2   0   0
                1   1   1

152)           -2   0   0
                1   0   0

153)           -2   0  -1 or 1
                1   0     0

154)           -2   0     0
                1   0  -1 or 1

154)(a-h)        -2   1   1      or   -2   0   0
                  1   0   0             1  -1   1
```

None of these gives new non-triangular faces.

****

```
151a)        -2    0    0      2   -2   -2
              1    1    1      3    3    3
              0   -1   -1     -4   -2   -2   This is (Tr5)
              0    0   -1     -2    2   -4
              0   -1    0     -2   -4    2

              0   -2    0     -4   -8    4
              1    1    1      3    3    3
             -1    0   -1     -1    1   -5   a,c not both null
              0    0   -1     -2    2   -4
             -1    0    0      1   -1   -1
```

****

```
151b)        -2    0    0      2   -2   -2
              1    1    1      3    3    3
              0   -1    0     -2   -4    2   a,c not both null
              0   -1    0     -2   -4    2
              0    0   -1     -2    2   -4
              0    0   -1     -2    2   -4

              0   -2    0     -4   -8    4
              1    1    1      3    3    3
             -1    0    0      1   -1   -1        ditto
             -1    0    0      1   -1   -1
              0    0   -1     -2    2   -4
              0    0   -1     -2    2   -4
```

****

152a)	-2	0	0	2	-2	-2	
	1	0	0	-1	1	1	
	0	1	1	4	2	2	
	0	-1	-1	-4	-2	-2	This is (Tr6)
	0	-1	0	-2	-4	2	
	0	0	-1	-2	2	-4	

	0	-2	0	-4	-8	4	
	0	1	0	2	4	-2	
	1	0	1	1	-1	5	a', c not both null
	-1	0	-1	-1	1	-5	
	-1	0	0	1	-1	-1	
	0	0	-1	-2	2	-4	

****

152b)	-2	0	0	2	-2	-2	
	1	0	0	-1	1	1	
	0	1	-1	0	6	-6	
	0	-1	1	0	-6	6	a,c not both null
	0	-1	0	-2	-4	2	
	0	0	-1	-2	2	-4	

	0	0	-2	-4	4	-8	
	0	0	1	2	-2	4	
	1	-1	0	-3	-3	3	ditto
	-1	1	0	3	3	-3	
	-1	0	0	1	-1	-1	
	0	-1	0	-2	-4	2	

****

152c)	-2	0	0	2	-2	-2	
	1	0	0	-1	1	1	
	0	1	-1	0	6	-6	This is (Tr7)
	0	-1	-1	-4	-2	-2	
	0	-1	0	-2	-4	2	
	0	0	1	2	-2	4	

	0	0	-2	-4	4	-8	
	0	0	1	2	-2	4	
	1	-1	0	-3	-3	3	a,c not both null
	-1	-1	0	-1	-5	1	
	-1	0	0	1	-1	-1	
	0	1	0	2	4	-2	

	0	0	-2	-4	4	-8	
	0	0	1	2	-2	4	
	-1	1	0	3	3	-3	ditto
	-1	-1	0	-1	-5	1	
	0	-1	0	-2	-4	2	
	1	0	0	-1	1	1	

****

152d)

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	-1	-1	-4	-2	-2	This is (Tr8)
0	-1	-1	-4	-2	-2	
0	1	0	2	4	-2	
0	0	1	2	-2	4	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	
-1	0	-1	-1	1	-1	a,c not both null
-1	0	-1	-1	1	-5	
1	0	0	-1	1	1	
0	0	1	2	-2	4	

****

152e)

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	1	1	4	2	2	
0	-1	0	-2	-4	2	This is (Tr9)
0	-1	0	-2	-4	2	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	
1	0	1	1	-1	5	
-1	0	0	1	-1	-1	a,c not both null
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

****

152f)

-2	0	0	2	-2	-2	
1	0	0	-1	1	1	
0	1	-1	0	6	-6	
0	-1	0	-2	-4	2	a,c not both null
0	-1	0	-2	-4	2	
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

0	-2	0	-4	-8	4	
0	1	0	2	4	-2	
1	0	-1	-3	3	-3	
-1	0	0	1	-1	-1	ditto
-1	0	0	1	-1	-1	
0	0	1	2	-2	4	
0	0	-1	-2	2	-4	

	0	-2	0	-4	-8	4	
	0	1	0	2	4	-2	
	-1	0	1	3	-3	3	
	0	0	-1	-2	2	-4	ditto
	0	0	-1	-2	2	-4	
	1	0	0	-1	1	1	
	-1	0	0	1	-1	-1	

****

152g)	-2	0	0	2	-2	-2	
	1	0	0	-1	1	1	
	0	-1	-1	-4	-2	-2	
	0	1	0	2	4	-2	This is (Tr10)
	0	-1	0	-2	-4	2	
	0	0	1	2	-2	4	
	0	0	-1	-2	2	-4	

	0	-2	0	-4	-8	4	
	0	1	0	2	4	-2	
	-1	0	-1	-1	1	-5	a,c not both null
	1	0	0	-1	1	1	
	-1	0	0	1	-1	-1	
	0	0	1	2	-2	4	
	0	0	-1	-2	2	-4	

****

152h)	-2	0	0	2	-2	-2	
	1	0	0	-1	1	1	
	0	1	0	2	4	-2	
	0	-1	0	-2	-4	2	a,c not both null
	0	-1	0	-2	-4	2	
	0	0	1	2	-2	4	
	0	0	-1	-2	2	-4	
	0	0	-1	-2	2	-4	

	0	-2	0	-4	-8	4	
	0	1	0	2	4	-2	
	1	0	0	-1	1	1	
	-1	0	0	1	-1	-1	ditto
	-1	0	0	1	-1	-1	
	0	0	1	2	-2	4	
	0	0	-1	-2	2	-4	
	0	0	-1	-2	2	-4	

****

153a)	0	-1	-2	-6	0	-6	
	0	0	1	2	-2	4	
	1	1	0	1	5	-1	a',c not both null
	-1	-1	0	-1	-5	1	
	-1	0	0	1	-1	-1	

	-1	0	-2	-3	3	-9	
	0	0	1	2	-2	4	ditto
	1	1	0	1	5	-1	

```
               -1   -1    0     -1   -5    1
                0   -1    0     -2   -4    2

                                 ***************

153b)           0   -1   -2     -6    0   -6
                0    0    1      2   -2    4
                1   -1    0     -3   -3    3   a',c not both null
               -1    1    0      3    3   -3
               -1    0    0      1   -1   -1

               -1    0   -2     -3    3   -9
                0    0    1      2   -2    4
               -1    1    0      3    3   -3         ditto
                1   -1    0     -3   -3    3
                0   -1    0     -2   -4    2

                                 ***************

153c)           0   -1   -2     -6    0   -6
                0    0    1      2   -2    4   a',c not both null
                1    0    0     -1    1    1
               -1   -1    0     -1   -5    1
               -1    1    0      3    3   -3

               -1    0   -2     -3    3   -9
                0    0    1      2   -2    4
                0    1    0      2    4   -2          ditto
               -1   -1    0     -1   -5    1
                1   -1    0     -3   -3    3

                                 ***************

153d)           0   -1   -2     -6    0   -6
                0    0    1      2   -2    4
                1    1    0      1    5   -1
               -1    0    0      1   -1   -1   a',c not both null
               -1    0    0      1   -1   -1
                0   -1    0     -2   -4    2

               -1    0   -2     -3    3   -9
                0    0    1      2   -2    4
                1    1    0      1    5   -1         ditto
                0   -1    0     -2   -4    2
                0   -1    0     -2   -4    2
               -1    0    0      1   -1   -1

                                 ***************

153e)           0   -1   -2     -6    0   -6
                0    0    1      2   -2    4
                1   -1    0     -3   -3    3   a', c not both null
               -1    0    0      1   -1   -1
               -1    0    0      1   -1   -1
                0    1    0      2    4   -2
```

```
           -1    0   -2     -3    3   -9
            0    0    1      2   -2    4
           -1    1    0      3    3   -3          ditto
            0   -1    0     -2   -4    2
            0   -1    0     -2   -4    2
            1    0    0     -1    1    1

                            ***************

153f)       0   -1   -2     -6    0   -6
            0    0    1      2   -2    4
            1    0    0     -1    1    1   a', c not both null
           -1    1    0      3    3   -3
           -1    0    0      1   -1   -1
            0   -1    0     -2   -4    2

           -1    0   -2     -3    3   -9
            0    0    1      2   -2    4
            0    1    0      2    4   -2          ditto
            1   -1    0     -3   -3    3
            0   -1    0     -2   -4    2
           -1    0    0      1   -1   -1

                            ***************

153g)       0   -1   -2     -6    0   -6
            0    0    1      2   -2    4
            1    0    0     -1    1    1   a',c not both null
           -1   -1    0     -1   -5    1
           -1    0    0      1   -1   -1
            0    1    0      2    4   -2

           -1    0   -2     -3    3   -9
            0    0    1      2   -2    4
            0    1    0      2    4   -2          ditto
           -1   -1    0     -1   -5    1
            0   -1    0     -2   -4    2
            1    0    0     -1    1    1

                            ***************

153h)       0   -1   -2     -6    0   -6
            0    0    1      2   -2    4
            1    0    0     -1    1    1
           -1    0    0      1   -1   -1   a',c not both null
           -1    0    0      1   -1   -1
            0    1    0      2    4   -2
            0   -1    0     -2   -4    2

           -1    0   -2     -3    3   -9
            0    0    1      2   -2    4
            0    1    0      2    4   -2
            0   -1    0     -2   -4    2          ditto
            0   -1    0     -2   -4    2
            1    0    0     -1    1    1
           -1    0    0      1   -1   -1
```

****

153aa)	0	1	-2	-2	8	-10	
	0	0	1	2	-2	4	
	1	-1	0	-3	-3	3	a',c not both null
	-1	-1	0	-1	-5	1	
	-1	0	0	1	-1	-1	
	1	0	-2	-5	5	-7	
	0	0	1	2	-2	4	
	-1	1	0	3	3	-3	ditto
	-1	-1	0	-1	-5	1	
	0	-1	0	-2	-4	2	

****

153bb)	0	1	-2	-2	8	-10	
	0	0	1	2	-2	4	
	-1	-1	0	-1	-5	1	a',c not both null
	-1	-1	0	-1	-5	1	
	1	0	0	-1	1	1	
	1	0	-2	-5	5	-7	
	0	0	1	2	-2	4	
	-1	-1	0	-1	-5	1	ditto
	-1	-1	0	-1	-5	1	
	0	1	0	2	4	-2	

****

153cc)	0	1	-2	-2	8	-10	
	0	0	1	2	-2	4	
	1	-1	0	-3	-3	3	a',c not both null
	0	-1	0	-2	-4	2	
	-1	0	0	1	-1	-1	
	-1	0	0	1	-1	-1	
	1	0	-2	-5	5	-7	
	0	0	1	2	-2	4	
	-1	1	0	3	3	-3	ditto
	-1	0	0	1	-1	-1	
	0	-1	0	-2	-4	2	
	0	-1	0	-2	-4	2	

****

153dd)	0	1	-2	-2	8	-10	
	0	0	1	2	-2	4	
	-1	-1	0	-1	-5	1	a',c not both null
	0	-1	0	-2	-4	2	
	1	0	0	-1	1	1	
	-1	0	0	1	-1	-1	
	1	0	-2	-5	5	-7	
	0	0	1	2	-2	4	
	-1	-1	0	-1	-5	1	
	-1	0	0	1	-1	-1	ditto

```
            0   1   0    2   4  -2
            0  -1   0   -2  -4   2

                       ***************

153ee)      0   1  -2   -2   8 -10
            0   0   1    2  -2   4
            0  -1   0   -2  -4   2
            0  -1   0   -2  -4   2   a',c not both null
           -1   0   0    1  -1  -1
           -1   0   0    1  -1  -1
            1   0   0   -1   1   1

            1   0  -2   -5   5  -7
            0   0   1    2  -2   4
           -1   0   0    1  -1  -1
           -1   0   0    1  -1  -1        ditto
            0  -1   0   -2  -4   2
            0  -1   0   -2  -4   2
            0   1   0    2   4  -2

                       ***************

154         0   0  -2   -4   4  -8
            0  -1   1    0  -6   6
            1   1   0    1   5  -1   a,c not both null
           -1  -1   0   -1  -5   1
           -1   0   0    1  -1  -1

            0   0  -2   -4   4  -8
           -1   0   1    3  -3   3
            1   1   0    1   5  -1        ditto
           -1  -1   0   -1  -5   1
            0  -1   0   -2  -4   2

                       ***************

            0   0  -2   -4   4  -8
            0  -1   1    0  -6   6
            1  -1   0   -3  -3   3   a,c not both null
           -1   1   0    3   3  -3
           -1   0   0    1  -1  -1

            0   0  -2   -4   4  -8
           -1   0   1    3  -3   3
           -1   1   0    3   3  -3        ditto
            1  -1   0   -3  -3   3
            0  -1   0   -2  -4   2

                       ***************

            0   0  -2   -4   4  -8
            0  -1   1    0  -6   6
            1   0   0   -1   1   1  a,c not both null
           -1  -1   0   -1  -5   1
           -1   1   0    3   3  -3
```

0	0	-2	-4	4	-8	
-1	0	1	3	-3	3	
0	1	0	2	4	-2	ditto
-1	-1	0	-1	-5	1	
1	-1	0	-3	-3	3	

****

0	0	-2	-4	4	-8	
0	-1	1	0	-6	6	
1	1	0	1	5	-1	
-1	0	0	1	-1	-1	a,c not both null
-1	0	0	1	-1	-1	
0	-1	0	-2	-4	2	

0	0	-2	-4	4	-8	
-1	0	1	3	-3	3	
1	1	0	1	5	-1	
0	-1	0	-2	-4	2	ditto
0	-1	0	-2	-4	2	
-1	0	0	1	-1	-1	

****

0	0	-2	-4	4	-8	
0	-1	1	0	-6	6	
1	-1	0	-3	-3	3	
-1	0	0	1	-1	-1	a,c not both null
-1	0	0	1	-1	-1	
0	1	0	2	4	-2	

0	0	-2	-4	4	-8	
-1	0	1	3	-3	3	
-1	1	0	3	3	-3	
0	-1	0	-2	-4	2	ditto
0	-1	0	-2	-4	2	
1	0	0	-1	1	1	

****

0	0	-2	-4	4	-8	
0	-1	1	0	-6	6	
1	0	0	-1	1	1	
-1	1	0	3	3	-3	a,c not both null
-1	0	0	1	-1	-1	
0	-1	0	-2	-4	2	

0	0	-2	-4	4	-8	
-1	0	1	3	-3	3	
0	1	0	2	4	-2	ditto
1	-1	0	-3	-3	3	
0	-1	0	-2	-4	2	
-1	0	0	1	-1	-1	

****

```
 0   0  -2     -4   4  -8
 0  -1   1      0  -6   6
 1   0   0     -1   1   1   a,c not both null
-1  -1   0     -1  -5   1
-1   0   0      1  -1  -1
 0   1   0      2   4  -2

 0   0  -2     -4   4  -8
-1   0   1      3  -3   3
 0   1   0      2   4  -2         ditto
-1  -1   0     -1  -5   1
 0  -1   0     -2  -4   2
 1   0   0     -1   1   1

               ***************

 0   0  -2     -4   4  -8
 0  -1   1      0  -6   6
 1   0   0     -1   1   1
-1   0   0      1  -1  -1   a,c not both null
-1   0   0      1  -1  -1
 0   1   0      2   4  -2
 0  -1   0     -2  -4   2

 0   0  -2     -4   4  -8
-1   0   1      3  -3   3
 0   1   0      2   4  -2
 0  -1   0     -2  -4   2         ditto
 0  -1   0     -2  -4   2
 1   0   0     -1   1   1
-1   0   0      1  -1  -1

               ***************

 0   0  -2     -4   4  -8
 0   1   1      4   2   2   orthogonality implies
 1  -1   0     -3  -3   3     d_5=1, contradicting
-1  -1   0     -1  -5   1     nullity
-1   0   0      1  -1  -1

 0   0  -2     -4   4  -8
 1   0   1      1  -1   5
-1   1   0      3   3  -3   a,c not both null
-1  -1   0     -1  -5   1
 0  -1   0     -2  -4   2

               ***************

 0   0  -2     -4   4  -8
 0   1   1      4   2   2   orthogonality fails
-1  -1   0     -1  -5   1
-1  -1   0     -1  -5   1
 1   0   0      1  -1  -1
```

```
 0   0  -2    -4   4  -8
 1   0   1     1  -1   5
-1  -1   0    -1  -5   1   a,c not both null
-1  -1   0    -1  -5   1
 0   1   0     2   4  -2
```

****

```
 0   0  -2    -4   4  -8
 0   1   1     4   2   2
 1  -1   0    -3  -3   3   orthogonality fails
 0  -1   0    -2  -4   2
-1   0   0     1  -1  -1
-1   0   0     1  -1  -1

 0   0  -2    -4   4  -8
 1   0   1     1  -1   5
-1   1   0     3   3  -3
-1   0   0     1  -1  -1   a,c not both null
 0  -1   0    -2  -4   2
 0  -1   0    -2  -4   2
```

****

```
 0   0  -2    -4   4  -8
 0   1   1     4   2   2
-1  -1   0    -1  -5   1   orthogonality fails
 0  -1   0    -2  -4   2
 1   0   0    -1   1   1
-1   0   0     1  -1  -1

 0   0  -2    -4   4  -8
 1   0   1     1  -1   5
-1  -1   0    -1  -5   1
-1   0   0     1  -1  -1   a,c not both null
 0   1   0     2   4  -2
 0  -1   0    -2  -4   2
```

****

```
 0   0  -2    -4   4  -8
 0   1   1     4   2   2
 0  -1   0    -2  -4   2
 0  -1   0    -2  -4   2   This is (Tr14)
-1   0   0     1  -1  -1
-1   0   0     1  -1  -1
 1   0   0    -1   1   1

 0   0  -2    -4   4  -8
 1   0   1     1  -1   5
-1   0   0     1  -1  -1
-1   0   0     1  -1  -1   a,c not both null
 0  -1   0    -2  -4   2
 0  -1   0    -2  -4   2
 0   1   0     2   4  -2
```

****

154a)

-2	-1	1	equivalent to 20)
1	0	0	
0	1	-1	
0	-1	-1	

****

154b)

-2	-1	1	equivalent to 129)
1	0	0	
0	-1	-1	
0	1	0	
0	0	-1	

****

154c) Triangle

-1	-2	1	-1	-11	7	
0	1	0	2	4	-2	
1	0	-1	-3	3	-3	a,c not both null
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	

1	-2	-1	-7	-5	1	
0	1	0	2	4	-2	
-1	0	1	3	-3	3	orthogonality fails
0	0	-1	-2	2	-4	
-1	0	0	1	-1	-1	

****

154d)

-1	-2	1	-1	-11	7	
0	1	0	2	4	-2	
1	0	0	-1	1	1	a,c not both null
-1	0	0	1	-1	-1	
0	0	-1	-2	2	-4	
0	0	-1	-2	2	-4	

****

154e)

-2	0	0	equivalent to 16)
1	-1	1	
0	-1	-1	
0	1	-1	

****

154f)

0	0	-2	-4	4	-8	
1	-1	1	-1	-5	7	
-1	-1	0	-1	-5	1	a,c not both null
0	1	0	2	4	-2	
-1	0	0	1	-1	-1	

0	0	-2	-4	4	-8	
-1	1	1	5	1	1	
-1	-1	0	-1	-5	1	orthogonality fails
1	0	0	-1	1	1	
0	-1	0	-2	-4	2	

****

154g)

0	0	-2	-4	4	-8	
1	-1	1	-1	-5	7	
0	-1	0	-2	-4	2	a,c not both null
-1	1	0	3	3	-3	
-1	0	0	1	-1	-1	

0	0	-2	-4	4	-8	
-1	1	1	5	1	1	
-1	0	0	1	-1	-1	orthogonality fails
1	-1	0	-3	-3	3	
0	-1	0	-2	-4	2	

****

154h)

0	0	-2	-4	4	-8	
1	-1	1	-1	-5	7	
0	-1	0	-2	-4	2	
0	1	0	2	4	-2	a, c not both null
-1	0	0	1	-1	-1	
-1	0	0	1	-1	-1	

0	0	-2	-4	4	-8	
-1	1	1	5	1	1	
-1	0	0	1	-1	-1	
1	0	0	-1	1	1	orthogonality fails
0	-1	0	-2	-4	2	
0	-1	0	-2	-4	2	

****

We have therefore concluded looking at 2-planes including a type III.

The 2-plane containing three type I is example 0).

****

155) 2-planes with a type II and two type I will include two type III so have already been covered.

****

Next consider 2-planes including two type II and a type I.

We first take the two type II to overlap in two places.

****

```
156)      1   1  -1       not face; this gives trapezium (T*3)
         -1  -1   0
         -1   0   0
          0  -1   0
```

****

```
157)      1   1   0      now   -1 -1  0  0   are also in the 2-plane
         -1  -1  -1            -1 -1 -1 -1
         -1   0   0             1  0 -1  1
          0  -1   0             0  1  1 -1
```

with equation $X_2= -1$, $X_1 + X_3 + X_4 = 0$, $X_i =0$ for $i > 4$.
This is the hexagon (H3).

If one vertex is present so is the opposite vertex. Moreover the centre must be present. So every subtriangle has an edge midpoint present and so is ruled out by Remarks 6.13, 6.14.

****

```
158)     1   1   0      equivalent to 33)
        -1  -1   0
        -1   0  -1
         0  -1   0
```

****

```
159)     1   1   0      now   0  0  are in 2-plane
        -1  -1   0            0  0
        -1   0   0           -1  1
         0  -1   0            1 -1
         0   0  -1           -1 -1
```

given by $X_1 + X_2 = 0$, $X_2 + X_5 = -1$, $X_1 + X_3 + X_4 = 0$, $X_i = 0 : i > 4$.
This is trapezium (T6).

```
          1      1      0       1      5     -1
         -1     -1      0      -1     -5      1
         -1      0      0       1     -1     -1     a,c not both null
          0     -1      0      -2     -4      2
          0      0     -1      -2      2     -4
```

****

```
160)      1  -1   0     equivalent to 33)
         -1   1   0
         -1   0  -1
          0  -1   0
```

****

```
161)      1  -1   0     now   0  is in the 2-plane.
         -1   1   0           0
         -1   0   0          -1
          0  -1   0          -1
          0   0  -1           1
```

This is parallelogram, but not a face.

```
 1     -1      0      -3     -3      3
-1      1      0       3      3     -3
-1      0      0       1     -1     -1      a,c not both null
 0     -1      0      -2     -4      2
 0      0     -1      -2      2     -4
```

****

```
162)      1  -1   0     equivalent to 157)
         -1  -1  -1
         -1   0   0
          0   1   0
```

****

```
163)      1  -1   0     equivalent to 20)
         -1  -1   0
         -1   0  -1
          0   1   0
```

****

164) Triangle

```
 1  -1   0     -3  -3   3
-1  -1   0     -1  -5   1
-1   0   0      1  -1  -1     a,c not both null
 0   1   0      2   4  -2
 0   0  -1     -2   2  -4

-1   1   0      3   3  -3
-1  -1   0     -1  -5   1
 0  -1   0     -2  -4   2           ditto
 1   0   0     -1   1   1
 0   0  -1     -2   2  -4
```

****

```
165)     -1  -1  -1     equivalent to 157)
         -1  -1   0
          1   0   0
          0   1   0
```

****

```
166)     -1  -1   0    now  0   0  are in 2-plane
         -1  -1   0         0   0
          1   0   0         1  -1
          0   1   0        -1   1
          0   0  -1        -1  -1
```

$X_1 = X_2$, $X_2 + X_5 = -1$, $X_1 + X_3 + X_4 = 0$, $X_i = 0 : i > 4$.

This is a trapezium, equivalent to 159) via the symmetry

```
(X_1, X_2, X_3, X_4, X_5) --> (-X_1, X_2, -X_4, -X_3, X_5).
```

****

Next consider when the two type II overlap in just one place.

****

```
167)    1  1  0       equivalent to 166)
       -1  0 -1
       -1  0  0
        0 -1  0
        0 -1  0
```

****

```
168)    Triangle

        1  0  1    1 -1  5
       -1  0  0    1  1 -1
       -1  0  0    1 -1 -1      a,c not both null
        0  0 -1   -2  2 -4
        0  0 -1   -2  2 -4
        0 -1  0   -2 -4  2

        1  0  1    1 -1  5
        0  0 -1   -2  2 -4
        0  0 -1   -2  2 -4            ditto
       -1  0  0    1 -1 -1
       -1  0  0    1 -1 -1
        0 -1  0   -2 -4  2
```

****

```
169)    1  0  0     now  0 -1  are in the 2-plane
       -1 -1 -1         -1 -1
       -1  0  0          0  1
        0 -1  0          1  0
        0  1  0         -1  0
```

```
X_2 = -1, X_1 + X_3 = 0, X_4 + X_5 = 0, X_i = 0: i > 4.
```

This is the square with midpoint (S). The centre must be present. If a vertex is present so is the opposite vertex. Hence we cannot obtain a subtriangle.

****

```
170)    1  0  0     equivalent to 159)
       -1 -1  0
       -1  0 -1
        0 -1  0
        0  1  0
```

****

```
171)    Triangle

         1  0  0   -1  1  1
        -1  0 -1   -1  1 -5
        -1  0  0    1 -1 -1
         0  0 -1   -2  2 -4      a,c not both null
         0  0  1    2 -2  4
         0 -1  0   -2 -4  2

         0  0  1    2 -2  4
        -1  0 -1   -1  1 -5
         0  0 -1   -2  2 -4
        -1  0  0    1 -1 -1            ditto
         1  0  0   -1  1  1
         0 -1  0   -2 -4  2
```

****

```
172)     1 -1  0
        -1  0 -1       equivalent to 159)
        -1  0  0
         0  1  0
         0  1  0
```

****

```
173)     1 -1  0
        -1  0  0      equivalent to 166)
        -1  0  0
         0  1  0
         0 -1 -1
```

****

```
174)    Triangle
         1   0  -1    -3   3  -3
        -1   0   0     1  -1  -1
        -1   0   0     1  -1  -1     a,c not both null
         0   0   1     2  -2   4
         0   0  -1    -2   2  -4
         0  -1   0    -2  -4   2

        -1   0   1     3  -3   3
         0   0  -1    -2   2  -4
         0   0  -1    -2   2  -4          ditto
         1   0   0    -1   1   1
        -1   0   0     1  -1  -1
         0  -1   0    -2  -4   2
```

****

Now consider no type II overlapping.

```
175)     1   0  -1
        -1   0   0     triangle, not face
        -1   0   0
         0   1   0
         0  -1   0
         0  -1   0
```

```
                    ****************

176)    1   0   0     now -1  is in the 2-plane
       -1   0  -1          -1
       -1   0   0           1
        0   1   0           0
        0  -1   0           0
        0  -1   0           0
```

This is a triangle with midpoint of one edge.

```
        1  0  0   -1  1  1
       -1 -1  0   -1 -5  1
       -1  0  0    1 -1 -1    a,c not both null
        0  0  1    2 -2  4
        0  0 -1   -2  2 -4
        0  0 -1   -2  2 -4

        0  0  1    2 -2  4
        0 -1 -1   -4 -2 -2
        0  0 -1   -2  2 -4         ditto
        1  0  0   -1  1  1
       -1  0  0    1 -1 -1
       -1  0  0    1 -1 -1
                                           ***************
```

```
177)    Triangle

        1  0  0   -1  1  1
       -1  0  0    1 -1 -1
       -1  0  0    1 -1 -1
        0  1  0    2  4 -2    a,c not both null
        0 -1  0   -2 -4  2
        0 -1  0   -2 -4  2
        0  0 -1   -2  2 -4
                                           ***************
```

Now consider spanning sets with three type II vectors.

We do not need to consider triples where two index sets are the same, as then there is a type I vector in the 2-plane, so this case will already have been dealt with. So we may assume all three index sets are distinct.

First consider cases where two index sets are disjoint.

We start with the case where the index set of the third vector is contained in the union of the index sets of the first two (all possible non-triangular examples are like this).

```
178          1  0 -1    -3  3 -3
            -1  0  1     3 -3  3
            -1  0  0     1 -1 -1     a,c not both null
             0  1 -1     0  6 -6
             0 -1  0    -2 -4  2
             0 -1  0    -2 -4  2

             0  1 -1     0  6 -6
             0 -1  1     0 -6  6
             0 -1  0    -2 -4  2          ditto
             1  0 -1    -3  3 -3
            -1  0  0     1 -1 -1
            -1  0  0     1 -1 -1

            -1  1  0     3  3 -3
             1 -1  0    -3 -3  3
             0 -1  0    -2 -4  2          ditto
            -1  0  1     3 -3  3
             0  0 -1    -2  2 -4
             0  0 -1    -2  2 -4
```

****

```
179)         1  0 -1   now   0  is in the 2-plane
            -1  0  1         0
            -1  0  0        -1
             0  1  0        -1
             0 -1 -1         0
             0 -1  0         1
```

This is parallelogram, but neither it nor its subtriangles are faces.

****

```
180)        1  0  1    now   0  is in the 2-plane
           -1  0 -1          0
           -1  0  0          1
            0  1 -1          0
            0 -1  0         -1
            0 -1  0         -1
```

This is the parallelogram (P8).

```
            1  0  1    1 -1  5
           -1  0 -1   -1  1 -5
           -1  0  0    1 -1 -1     a,c not both null
            0  1 -1    0  6 -6
            0 -1  0   -2 -4  2
            0 -1  0   -2 -4  2

            0  1  1    4  2  2
            0 -1 -1   -4 -2 -2
            0 -1  0   -2  4  2     a', c not both null
            1  0 -1   -3  3 -3
           -1  0  0    1 -1 -1
           -1  0  0    1 -1 -1
```

```
 1  1  0    1  5 -1
-1 -1  0   -1 -5  1
 0 -1  0   -2 -4  2          ditto
-1  0  1    3 -3  3
 0  0 -1   -2  2 -4
 0  0 -1   -2  2 -4

 1  0  0   -1  1  1
-1  0  0    1 -1 -1
-1  0  1    3 -3  3     a',c not both null
 0  1  0    2  4 -2
 0 -1 -1   -4 -2 -2
 0 -1 -1   -4 -2 -2

 0  1  0    2  4 -2
 0 -1  0   -2 -4  2
 0 -1  1    0 -6  6          ditto
 1  0  0   -1  1  1
-1  0 -1   -1  1 -5
-1  0 -1   -1  1 -5

 0  1  0    2  4 -2
 0 -1  0   -2 -4  2
 1 -1  0   -3 -3  3
 0  0  1    2 -2  4          ditto
-1  0 -1   -1  1 -5
-1  0 -1   -1  1 -5
```

****

```
181)        1  0  1     now   0    is in the 2-plane
           -1  0 -1           0
           -1  0  0          -1
            0  1  0           1
            0 -1 -1           0
            0 -1  0          -1
```

This is the parallelogram (P9).

```
 1  0 -1    1 -1  5
-1  0 -1   -1  1 -5
-1  0  0    1 -1 -1    orthogonality fails
 0  1  0    2  4 -2
 0 -1 -1   -4 -2 -2
 0 -1  0   -2 -4  2

 0  1  1    4  2  2
 0 -1 -1   -4 -2 -2
 0 -1  0   -2 -4  2          ditto
 1  0  0   -1  1  1
-1  0 -1   -1  1 -5
-1  0  0    1 -1 -1
```

```
 1  1  0    1  5 -1
-1 -1  0   -1 -5  1    orthogonality implies
 0 -1  0   -2 -4  2     d_1 = d_2 = d_5 = 1
 0  0  1    2 -2  4       so a' not null
-1  0 -1   -1  1 -5
 0  0 -1   -2  2 -4
```

****

182) Triangle

```
 1  0  0   -1  1  1
-1  0  1    3 -3  3
-1  0 -1   -1  1 -5    a,c not both null
 0  1 -1    0  6 -6
 0 -1  0   -2 -4  2
 0 -1  0   -2 -4  2

 0  1  0    2  4 -2
 0 -1  1    0 -6  6
 0 -1 -1   -4 -2 -2    a,a' not both null
 1  0 -1   -3  3 -3
-1  0  0    1 -1 -1
-1  0  0    1 -1 -1

 0  1  0    2  4 -2
 1 -1  0   -3 -3  3
-1 -1  0   -1 -5  1    a,c not both null
-1  0  1    3 -3  3
 0  0 -1   -2  2 -4
 0  0 -1   -2  2 -4
```

****

183) Triangle

```
 1  0  0   -1  1  1
-1  0  1    3 -3  3    orthogonality implies d_2 = 1,
-1  0 -1   -1  1 -5      so a not null
 0  1  0    2  4 -2
 0 -1 -1   -4 -2 -2
 0 -1  0   -2 -4  2

 0  1  0    2  4 -2
 0 -1  1    0 -6  6
 0 -1 -1   -4 -2 -2    orthogonality fails
 1  0  0   -1  1  1
-1  0 -1   -1  1 -5
-1  0  0    1 -1 -1

 0  1  0    2  4 -2
 1 -1  0   -3 -3  3
-1 -1  0   -1 -5  1    a,c not both null
 0  0  1    2 -2  4
-1  0 -1   -1  1 -5
 0  0 -1   -2  2 -4
```

****

```
184)      Triangle

           1  0 -1   -3  3 -3
          -1  0 -1   -1  1 -5
          -1  0  0    1 -1 -1   orthogonality implies d_1=1
           0  1  1    4  2  2     so a not null
           0 -1  0   -2 -4  2
           0 -1  0   -2 -4  2

           0  1 -1    0  6 -6
           0 -1 -1   -4 -2 -2
           0 -1  0   -2 -4  2   orthogonality fails
           1  0  1    1 -1  5
          -1  0  0    1 -1 -1
          -1  0  0    1 -1 -1

          -1  1  0    3  3 -3
          -1 -1  0   -1 -5  1
           0 -1  0   -2 -4  2   a,c not both null
           1  0  1    1 -1  5
           0  0 -1   -2  2 -4
           0  0 -1   -2  2 -4
```

****

```
185)      Triangle

           1  0 -1   -3  3 -3
          -1  0 -1   -1  1 -5
          -1  0  0    1 -1 -1   a,c not both null
           0  1  0    2  4 -2
           0 -1  1    0 -6  6
           0 -1  0   -2 -4  2

           0  1 -1    0  6 -6
           0 -1 -1   -4 -2 -2
           0 -1  0   -2 -4  2   a, a' not both null
           1  0  0   -1  1  1
          -1  0  1    3 -3  3
          -1  0  0    1 -1 -1

          -1  1  0    3  3 -3
          -1 -1  0   -1 -5  1
           0 -1  0   -2 -4  2   a,c not both null
           0  0  1    2 -2  4
           1  0 -1   -3  3 -3
           0  0 -1   -2  2 -4
```

****

```
186)       1  0  0    now  1   is also in 2-plane
          -1  0 -1         0
          -1  0 -1         0
           0  1  1         0
           0 -1  0        -1
           0 -1  0        -1
```

This is parallelogram (P10).

```
  1  0  0   -1  1  1
 -1  0 -1   -1  1 -5
 -1  0 -1   -1  1 -5   orthogonality fails
  0  1  1    4  2  2
  0 -1  0   -2 -4  2
  0 -1  0   -2 -4  2

  0  1  0    2  4 -2
  0 -1 -1   -4 -2 -2
  0 -1 -1   -4 -2 -2         ditto
  1  0  1    1 -1  5
 -1  0  0    1 -1 -1
 -1  0  0    1 -1 -1

  0  1  0    2  4 -2
 -1 -1  0   -1 -5  1
 -1 -1  0   -1 -5  1   orthogonality implies
  1  0  1    1 -1  5     d_4 =1 so a' not null
  0  0 -1   -2  2 -4
  0  0 -1   -2  2 -4
```

****

```
187)       1  0  0        equivalent to 180)
          -1  0 -1
          -1  0 -1
           0  1  0
           0 -1  1
           0 -1  0
```

****

Now we consider examples where two index sets are disjoint and the index set of the third vector is not contained in the union of the index sets of the first two vectors. All such examples are triangles.

```
187a)     1  0  1   1 -1  5
         -1  0 -1  -1  1 -5
         -1  0  0   1 -1 -1
          0  1  0   2  4 -2    a,c not both null
          0 -1  0  -2 -4  2
          0 -1  0  -2 -4  2
          0  0 -1  -2  2 -4

          0  1  1   4  2  2
          0 -1 -1  -4 -2 -2
          0 -1  0  -2 -4  2
          1  0  0  -1  1  1    This is (Tr15)
         -1  0  0   1 -1 -1
         -1  0  0   1 -1 -1
          0  0 -1  -2  2 -4
```

****

```
187b)     1  0 -1  -3  3 -3
         -1  0  1   3 -3  3
         -1  0  0   1 -1 -1
          0  1  0   2  4 -2    a,c not both null
          0 -1  0  -2 -4  2
          0 -1  0  -2 -4  2
          0  0 -1  -2  2 -4

          0  1 -1   0  6 -6
          0 -1  1   0 -6  6
          0 -1  0  -2 -4  2
          1  0  0  -1  1  1         ditto
         -1  0  0   1 -1 -1
         -1  0  0   1 -1 -1
          0  0 -1  -2  2 -4
                                ***************

187c)     1  0  0  -1  1  1
         -1  0 -1  -1  1 -5
         -1  0  1   3 -3  3
          0  1  0   2  4 -2    a,c not both null
          0 -1  0  -2 -4  2
          0 -1  0  -2 -4  2
          0  0 -1  -2  2 -4

          0  1  0   2  4 -2
          0 -1 -1  -4 -2 -2
          0 -1  1   0 -6  6
          1  0  0  -1  1  1    This is (Tr16)
         -1  0  0   1 -1 -1
         -1  0  0   1 -1 -1
          0  0 -1  -2  2 -4

          0  1  0   2  4 -2
         -1 -1  0  -1 -5  1
          1 -1  0  -3 -3  3
          0  0  1   2 -2  4    a,c not both null
          0  0 -1  -2  2 -4
          0  0 -1  -2  2 -4
         -1  0  0   1 -1 -1
                                ***************

187d)     1  0  0  -1  1  1
         -1  0 -1  -1  1 -5
         -1  0 -1  -1  1 -5
          0  1  0   2  4 -2    a,c not both null
          0 -1  0  -2 -4  2
          0 -1  0  -2 -4  2
          0  0  1   2 -2  4

          0  1  0   2  4 -2
          0 -1 -1  -4 -2 -2
          0 -1 -1  -4 -2 -2
          1  0  0  -1  1  1    This is (Tr17)
         -1  0  0   1 -1 -1
         -1  0  0   1 -1 -1
          0  0 -1  -2  2 -4
```

```
                         ***************

187e)     1  0  1   1 -1  5
         -1  0  0   1 -1 -1
         -1  0  0   1 -1 -1
          0  1 -1   0  6 -6    a,c not both null
          0 -1  0  -2 -4  2
          0 -1  0  -2 -4  2
          0  0 -1  -2  2 -4

          0  1  1   4  2  2
          0 -1  0  -2 -4  2
          0 -1  0  -2 -4  2
          1  0 -1  -3  3 -3    orthogonality fails
         -1  0  0   1 -1 -1
         -1  0  0   1 -1 -1
          0  0 -1  -2  2 -4

          1  1  0   1  5 -1
          0 -1  0  -2 -4  2
          0 -1  0  -2 -4  2
         -1  0  1   3 -3  3    a,c not both null
          0  0 -1  -2  2 -4
          0  0 -1  -2  2 -4
         -1  0  0   1 -1 -1

                         ***************

187f)     1  0  1   1 -1  5
         -1  0  0   1 -1 -1
         -1  0  0   1 -1 -1
          0  1  0   2  4 -2    orthogonality fails
          0 -1 -1  -4 -2 -2
          0 -1  0  -2 -4  2
          0  0 -1  -2  2 -4

          0  1  1   4  2  2
          0 -1  0  -2 -4  2
          0 -1  0  -2 -4  2
          1  0  0  -1  1  1          ditto
         -1  0 -1  -1  1 -5
         -1  0  0   1 -1 -1
          0  0 -1  -2  2 -4

          1  1  0   1  5 -1
          0 -1  0  -2 -4  2
          0 -1  0  -2 -4  2
          0  0  1   2 -2  4    a,c not both null
         -1  0 -1  -1  1 -5
          0  0 -1  -2  2 -4
         -1  0  0   1 -1 -1

                         ***************
```

```
187g)      1  0  0  -1  1  1
          -1  0  1   3 -3  3
          -1  0  0   1 -1 -1
           0  1  0   2  4 -2    orthogonality fails
           0 -1 -1  -4 -2 -2
           0 -1  0  -2 -4  2
           0  0 -1  -2  2 -4

           0  1  0   2  4 -2
           0 -1  1   0 -6  6
           0 -1  0  -2 -4  2
           1  0  0  -1  1  1    a,c not both null
          -1  0 -1  -1  1 -5
          -1  0  0   1 -1 -1
           0  0 -1  -2  2 -4

           0  1  0   2  4 -2
           1 -1  0  -3 -3  3
           0 -1  0  -2 -4  2
           0  0  1   2 -2  4          ditto
          -1  0 -1  -1  1 -5
           0  0 -1  -2  2 -4
          -1  0  0   1 -1 -1
                                 ***************

187gg)     1  0  0   -1  1  1
          -1  0  1    3 -3  3
          -1  0  0    1 -1 -1
           0  1 -1    0  6 -6   a,c not both null
           0 -1  0   -2 -4  2
           0 -1  0   -2 -4  2
           0  0 -1   -2  2 -4

           0  1  0    2  4 -2
           0 -1  1    0 -6  6
           0 -1  0   -2 -4  2
           1  0 -1   -3  3 -3         ditto
          -1  0  0    1 -1 -1
          -1  0  0    1 -1 -1
           0  0 -1   -2  2 -4

           0  1  0    2  4 -2
           1 -1  0   -3 -3  3
           0 -1  0   -2 -4  2
          -1  0  1    3 -3  3         ditto
           0  0 -1   -2  2 -4
           0  0 -1   -2  2 -4
          -1  0  0    1 -1 -1
                                 ***************

187h)     1  0 -1   -3  3 -3
         -1  0  0    1 -1 -1
         -1  0  0    1 -1 -1
          0  1 -1    0  6 -6     a,c not both null
          0 -1  0   -2 -4  2
          0 -1  0   -2 -4  2
          0  0  1    2 -2  4
```

```
         0  1 -1    0  6 -6
         0 -1  0   -2 -4  2
         0 -1  0   -2 -4  2
         1  0 -1   -3  3 -3         ditto
        -1  0  0    1 -1 -1
        -1  0  0    1 -1 -1
         0  0  1    2 -2  4

        -1  1  0    3  3 -3
         0 -1  0   -2 -4  2
         0 -1  0   -2 -4  2
        -1  0  1    3 -3  3         ditto
         0  0 -1   -2  2 -4
         0  0 -1   -2  2 -4
         1  0  0   -1  1  1

                               ***************

187i)    1  0 -1   -3  3 -3
        -1  0  0    1 -1 -1
        -1  0  0    1 -1 -1
         0  1  0    2  4 -2     orthogonality fails
         0 -1 -1   -4 -2 -2
         0 -1  0   -2 -4  2
         0  0  1    2 -2  4

         0  1 -1    0  6 -6
         0 -1  0   -2 -4  2
         0 -1  0   -2 -4  2
         1  0  0   -1  1  1     a,c not both null
        -1  0 -1   -1  1 -5
        -1  0  0    1 -1 -1
         0  0  1    2 -2  4

        -1  1  0    3  3 -3
         0 -1  0   -2 -4  2
         0 -1  0   -2 -4  2
         0  0  1    2 -2  4         ditto
        -1  0 -1   -1  1 -5
         0  0 -1   -2  2 -4
         1  0  0   -1  1  1

                               ***************

187j)    1  0  0   -1  1  1
        -1  0 -1   -1  1 -5
        -1  0  0    1 -1 -1
         0  1  0    2  4 -2     orthogonality fails
         0 -1 -1   -4 -2 -2
         0 -1  0   -2 -4  2
         0  0  1    2 -2  4
```

```
         0  0  1    2 -2  4
        -1  0 -1   -1  1 -5
         0  0 -1   -2  2 -4
         0  1  0    2  4 -2     a,c not both null
        -1 -1  0   -1 -5  1
         0 -1  0   -2 -4  2
         1  0  0   -1  1  1

                                ***************

187k)    1  0  1    1 -1  5
        -1  0  0    1 -1 -1
        -1  0  0    1 -1 -1
         0  1  0    2  4 -2     a,c not both null
         0 -1  0   -2 -4  2
         0 -1  0   -2 -4  2
         0  0 -1   -2  2 -4
         0  0 -1   -2  2 -4

         0  1  1    4  2  2
         0 -1  0   -2 -4  2
         0 -1  0   -2 -4  2
         1  0  0   -1  1  1     This is (Tr18)
        -1  0  0    1 -1 -1
        -1  0  0    1 -1 -1
         0  0 -1   -2  2 -4
         0  0 -1   -2  2 -4

         1  1  0    1  5 -1
         0 -1  0   -2 -4  2
         0 -1  0   -2 -4  2     a,c not both null
         0  0  1    2 -2  4
         0  0 -1   -2  2 -4
         0  0 -1   -2  2 -4
        -1  0  0    1 -1 -1
        -1  0  0    1 -1 -1

                                ***************

187l)    1  0  0   -1  1  1
        -1  0  1    3 -3  3
        -1  0  0    1 -1 -1
         0  1  0    2  4 -2     a,c not both null
         0 -1  0   -2 -4  2
         0 -1  0   -2 -4  2
         0  0 -1   -2  2 -4
         0  0 -1   -2  2 -4

         0  1  0    2  4 -2
         0 -1  1    0 -6  6
         0 -1  0   -2 -4  2
         1  0  0   -1  1  1          ditto
        -1  0  0    1 -1 -1
        -1  0  0    1 -1 -1
         0  0 -1   -2  2 -4
         0  0 -1   -2  2 -4
```

```
         0  1  0     2  4 -2
         1 -1  0    -3 -3  3
         0 -1  0    -2 -4  2
         0  0  1     2 -2  4          ditto
         0  0 -1    -2  2 -4
         0  0 -1    -2  2 -4
        -1  0  0     1 -1 -1
        -1  0  0     1 -1 -1
```

****

```
187m)    1  0  0    -1  1  1
        -1  0 -1    -1  1 -5
        -1  0  0     1 -1 -1
         0  1  0     2  4 -2     a,c not both null
         0 -1  0    -2 -4  2
         0 -1  0    -2 -4  2
         0  0 -1    -2  2 -4
         0  0  1     2 -2  4

         0  1  0     2  4 -2
         0 -1 -1    -4 -2 -2
         0 -1  0    -2 -4  2
         1  0  0    -1  1  1     This is (Tr19)
        -1  0  0     1 -1 -1
        -1  0  0     1 -1 -1
         0  0 -1    -2  2 -4
         0  0  1     2 -2  4

         0  1  0     2  4 -2
        -1 -1  0    -1 -5  1
         0 -1  0    -2 -4  2
         0  0  1     2 -2  4     a,c not both null
         0  0 -1    -2  2 -4
         0  0 -1    -2  2 -4
        -1  0  0     1 -1 -1
         1  0  0    -1  1  1
```

****

```
187n)    1  0  0    -1  1  1
        -1  0  0     1 -1 -1
        -1  0  0     1 -1 -1
         0  1  0     2  4 -2
         0 -1  0    -2 -4  2     a,c not both null
         0 -1  0    -2 -4  2
         0  0  1     2 -2  4
         0  0 -1    -2  2 -4
         0  0 -1    -2  2 -4
```

****

We can now assume that none of the index sets are disjoint. Recall also we may assume no two index sets are identical.

Possible index sets, up to permutation, are therefore:

```
123    123
 234    234
12 4    23 5
```

if each two index sets overlap in two places, and

```
123        123      123         123          123
  345        345      345         345          345
 2 4 6     12 4       3  67      234            34 6
```

if some pair of index sets overlap in just one place.

****

We consider triples with index sets

```
123
 234
12 4
```

```
188)        1   0   1       equivalent to 157)
           -1  -1  -1
           -1  -1   0
            0   1  -1
```

****

```
189)        1   0  -1       equivalent to 38)
           -1  -1   1
           -1  -1   0
            0   1  -1
```

****

```
190)        1   0   1       equivalent to 157)
           -1   1   0
           -1  -1  -1
            0  -1  -1
```

****

```
191)        1   0  -1       equivalent to 93)
           -1   1   0
           -1  -1   1
            0  -1  -1
```

****

```
192)        1   0  -1       equivalent to 157)
           -1   1   0
           -1  -1  -1
            0  -1   1
```

****

```
193)        1   0   1       equivalent to 40)
           -1   1  -1
           -1  -1   0
            0  -1  -1
```

```
                              ***************

194)       1   0  -1
          -1   1  -1       equivalent to 40)
          -1  -1   0
           0  -1   1

                              ***************

195)       1   0  -1       equivalent to 156)
          -1   1   1
          -1  -1   0
           0  -1  -1

                              ***************

196)      -1   0  -1       equivalent to 157)
          -1  -1  -1
           1   1   0
           0  -1   1

                              ***************

197)      -1   0  -1       equivalent to 40)
          -1  -1   1
           1   1   0
           0  -1  -1

                              ***************

198)      -1   0  -1       equivalent to 12)
          -1  -1   0
           1   1   1
           0  -1  -1

                              ***************

199)      -1   0   1
          -1  -1   0       equivalent to 156)
           1   1  -1
           0  -1  -1

                              ***************

200)      -1   0   1       equivalent to 93)
          -1   1  -1
           1  -1   0
           0  -1  -1

                              ***************

201)      -1   0  -1       equivalent to 156)
          -1   1   1
           1  -1   0
           0  -1  -1

                              ***************
```

Now we consider triples with index sets

```
                             123
                              234
                              23 5
```

202)   Triangle

```
      1    0    0      -1    1    1
     -1   -1   -1      -3   -3   -3
     -1   -1   -1      -3   -3   -3    a,c not both null
      0    1    0       2    4   -2
      0    0    1       2   -2    4
```

****

203)   Triangle

```
      1    0    0      -1    1    1
     -1   -1    1       1   -7    5
     -1   -1   -1      -3   -3   -3    a,c not both null
      0    1    0       2    4   -2
      0    0   -1      -2    2   -4

      0    0    1       2   -2    4
      1   -1   -1      -5   -1   -1    orthogonality implies
     -1   -1   -1      -3   -3   -3      d_3 = 1 so a' is not null
      0    1    0       2    4   -2
     -1    0    0       1   -1   -1
```

****

204)   Triangle

```
      1    0    0      -1    1    1
     -1   -1   -1      -3   -3   -3   orthogonality implies
     -1    1    1       5    1    1     d_2 =1, so a not null
      0   -1    0      -2   -4    2
      0    0   -1      -2    2   -4


      0    1    0       2    4   -2
     -1   -1   -1      -3   -3   -3
      1   -1    1      -1   -5   -7   a,c not both null
     -1    0    0       1   -1   -1
      0    0   -1      -2    2   -4
```

****

```
205)           1  0  0        triangle, not face
              -1 -1  1
              -1  1 -1
               0 -1  0
               0  0 -1
```

****

```
206)          -1  0  0       triangle, not face
               1  1 -1
              -1 -1  1
               0 -1  0
               0  0 -1
```

****

```
207)          -1  0  0       triangle, not face
               1 -1 -1
              -1  1  1
               0 -1  0
               0  0 -1
```

****

```
208)          -1    0    0      1   -1   -1
               1    1    1      3    3    3
              -1   -1   -1     -3   -3   -3   a,c not both null
               0   -1    0     -2   -4    2
               0    0   -1     -2    2   -4
```

****

Next we consider triples with index sets

```
123
  345
 2 4 6
```

i.e. each index set overlaps in exactly one place. All such examples give triangles except those equivalent to 209).

First consider cases where two vectors overlap with a 1 in the overlapping place.

```
-1   0   0    1  -1  -1
-1   0   1    3  -3   3
 1   1   0    1   5  -1     orthogonality fails
 0  -1  -1   -4  -2  -2
 0  -1   0   -2  -4   2
 0   0  -1   -2   2  -4

 0  -1   0   -2  -4   2
 0  -1   1    0  -6   6
 1   1   0    1   5  -1     a,c not both null
-1   0  -1   -1   1  -5
-1   0   0    1  -1  -1
 0   0  -1   -2   2  -4

 0  -1   0   -2  -4   2
 1  -1   0   -3  -3   3
 0   1   1    4   2   2     orthogonality fails
-1   0  -1   -1   1  -5
 0   0  -1   -2   2  -4
-1   0   0    1  -1  -1
```

****

```
-1  0  0    1  -1  -1
-1  0 -1   -1   1  -5
 1  1  0    1   5  -1    orthogonality implies
 0 -1 -1   -4  -2  -2    d_2 = 1 or 2 so a' not null
 0 -1  0   -2  -4   2
 0  0  1    2  -2   4

 0 -1  0   -2  -4   2
 0 -1 -1   -4  -2  -2
 1  1  0    1   5  -1    as above using d_3 and a
-1  0 -1   -1   1  -5
-1  0  0    1  -1  -1
 0  0  1    2  -2   4

 0 -1  0   -2  -4   2
-1 -1  0   -1  -5   1
 0  1  1    4   2   2    as above using d_2 and a
-1  0 -1   -1   1  -5
 0  0 -1   -2   2  -4
 1  0  0   -1   1   1
```

****

Next consider cases where no two vectors overlap with 1 in an overlapping place, but two vectors do overlap with -1 in the overlapping place.

```
-1  0  0    1  -1  -1
 1  0 -1   -3   3  -3
-1 -1  0   -1  -5   1    a,c not both null
 0  1 -1    0   6  -6
 0 -1  0   -2  -4   2
 0  0  1    2  -2   4

 0 -1  0   -2  -4   2
 0  1 -1    0   6  -6
-1 -1  0   -1  -5   1           ditto
 1  0 -1   -3   3  -3
-1  0  0    1  -1  -1
 0  0  1    2  -2   4

 0 -1  0   -2  -4   2
-1  1  0    3   3  -3
 0 -1 -1   -4  -2  -2   orthogonality fails
-1  0  1    3  -3   3
 0  0 -1   -2   2  -4
 1  0  0   -1   1   1
```

****

```
-1  0 -1   -1   1  -5
 1  0  0   -1   1   1
-1 -1  0   -1  -5   1   orthogonality implies
 0  1  0    2   4  -2    d_1 = 2, d_2 = 1 so a' not null
 0 -1 -1   -4  -2  -2
 0  0  1    2  -2   4
```

```
-1 -1  0   -1  -5   1
 0  1  0    2   4  -2
 0 -1 -1   -4  -2  -2   orthogonality implies d_5=2,
 0  0  1    2  -2   4    d_6 = 1 so a' not null
-1  0 -1   -1   1  -5
 1  0  0   -1   1   1

                    ***************
-1  0 -1
 1  0  0
-1 -1  0     This is equivalent to 209) (see below)
 0  1 -1
 0 -1  0
 0  0 -1

                    ***************

-1  0 -1   -1   1  -5
 1  0  0   -1   1   1
-1 -1  0   -1  -5   1    a,c not both null
 0  1  0    2   4  -2
 0 -1  1    0  -6   6
 0  0 -1   -2   2  -4

 0 -1 -1   -4  -2  -2
 0  1  0    2   4  -2
-1 -1  0   -1  -5   1    orthogonality fails
 1  0  0   -1   1   1
-1  0  1    3  -3   3
 0  0 -1   -2   2  -4

-1 -1  0   -1  -5   1
 0  1  0    2   4  -2
 0 -1 -1   -4  -2  -2           ditto
 0  0  1    2  -2   4
 1  0 -1   -3   3  -3
-1  0  0    1  -1  -1

                   ***************

-1  0  0    1  -1  -1
 1  0 -1   -3   3  -3
-1 -1  0   -1  -5   1    orthogonality fails
 0  1  0    2   4  -2
 0 -1  1    0  -6   6
 0  0 -1   -2   2  -4

 0 -1  0   -2  -4   2
 0  1 -1    0   6  -6
-1 -1  0   -1  -5   1    a,c not both null
 1  0  0    3   1   1
-1  0  1   -1  -3   3
 0  0 -1   -2   2  -4
```

```
 0 -1  0   -2  -4   2
-1  1  0    3   3  -3
 0 -1 -1   -4  -2  -2    orthogonality fails
 0  0  1    2  -2   4
 1  0 -1   -3   3  -3
-1  0  0    1  -1  -1
```

****

Finally, consider the case when no two vectors overlap with the same entry in the overlapping place.

```
-1  0  0    1  -1  -1
-1  0  1    3  -3   3
 1 -1  0   -3  -3   3    a,c not both null
 0  1 -1    0   6  -6
 0 -1  0   -2  -4   2
 0  0 -1   -2   2  -4

 0 -1  0   -2  -4   2
 0 -1  1    0  -6   6
-1  1  0    3   3  -3          ditto
 1  0 -1   -3   3  -3
-1  0  0    1  -1  -1
 0  0 -1   -2   2  -4

 0 -1  0   -2  -4   2
 1 -1  0   -3  -3   3
 0  1 -1    0   6  -6          ditto
-1  0  1    3  -3   3
 0  0 -1   -2   2  -4
-1  0  0    1  -1  -1
```

****

```
209)          1  0  0      now 1  is in the 2-plane
             -1  0 -1          0
             -1  1  0          0
              0 -1 -1          0
              0 -1  0         -1
              0  0  1         -1
```

This is parallelogram (P11). Subtriangles are

```
 1  0  0   -1   1   1
-1  0 -1   -1   1  -5
-1  1  0    3   3  -3    orthogonality fails
 0 -1 -1   -4  -2  -2
 0 -1  0   -2  -4   2
 0  0  1    2  -2   4

 0  1  0    2   4  -2
 0 -1 -1   -4  -2  -2
 1 -1  0   -3  -3   3           ditto
-1  0 -1   -1   1  -5
-1  0  0    1  -1  -1
 0  0  1    2  -2   4
```

```
 0  1  0    2   4  -2
-1 -1  0   -1  -5   1
 0 -1  1    0  -6   6    a,c not both null
-1  0 -1   -1   1  -5
 0  0 -1   -2   2  -4
 1  0  0   -1   1   1
```

****

```
 1  0  1    1  -1   5
-1  0  0    1  -1  -1
-1  1  0    3   3  -3    orthogonality fails
 0 -1  0   -2  -4   2
 0 -1 -1   -4  -2  -2
 0  0 -1   -2   2  -4

 0  1  1    4   2   2
 0 -1  0   -2  -4   2
 1 -1  0   -3  -3   3           ditto
-1  0  0    1  -1  -1
-1  0 -1   -1   1  -5
 0  0 -1   -2   2  -4

 1  1  0    1   5  -1
 0 -1  0   -2  -4   2
 0 -1  1    0  -6   6    a,c not both null
 0  0 -1   -2   2  -4
-1  0 -1   -1   1  -5
-1  0  0    1  -1  -1
```

****

Next we consider triples with index sets

```
123
  345
  3  67
```

These all give triangles. There are four possible triangles, depending on whether the entries in place 3 are {1,1,1}, {-1,-1,-1},{1,1,-1} or {-1,-1,1}.

```
{1,1,1}        -1  0  0   1  1 -1
               -1  0  0   1 -1 -1
                1  1  1   3  3  3
                0 -1  0  -2 -4  2     a,c not both null
                0 -1  0  -2 -4  2
                0  0 -1  -2  2 -4
                0  0 -1  -2  2 -4

{-1,-1,-1}      1  0  0  -1  1  1
               -1  0  0   1 -1 -1
               -1 -1 -1  -3 -3 -3
                0 -1  0  -2 -4  2          ditto
                0  1  0   2  4 -2
                0  0 -1  -2  2 -4
                0  0  1   2 -2  4
```

```
 {1,1,-1}       -1  0  0   1 -1 -1
                -1  0  0   1 -1 -1
                 1  1 -1  -1  7 -5
                 0 -1  0  -2 -4  2          ditto
                 0 -1  0  -2 -4  2
                 0  0  1   2 -2  4
                 0  0 -1  -2  2 -4

                 0  0 -1  -2  2 -4
                 0  0 -1  -2  2 -4
                -1  1  1   5  1  1
                 0 -1  0  -2 -4  2     orthogonality fails
                 0 -1  0  -2 -4  2
                 1  0  0  -1  1  1
                -1  0  0   1 -1 -1

{-1,-1,1}        1  0  0  -1  1  1
                -1  0  0   1 -1 -1
                -1 -1  1   1 -7  5
                 0  1  0   2  4 -2     a, c not both null
                 0 -1  0  -2 -4  2
                 0  0 -1  -2  2 -4
                 0  0 -1  -2  2 -4

                 0  0  1   2 -2  4
                 0  0 -1  -2  2 -4
                 1 -1 -1  -5 -1 -1
                 0  1  0   2  4 -2     orthogonality fails
                 0 -1  0  -2 -4  2
                -1  0  0   1 -1 -1
                -1  0  0   1 -1 -1
```

****

Next we consider triples with index sets

```
123
  345
  34 6
```

These all give triangles. We arrange the examples according to the entries in place 3. Within each case, we arrange them according to entries in place 4.

```
(1,1,1)        -1  0  0    1  -1  -1
               -1  0  0    1  -1  -1
                1  1  1    3   3   3   This is (Tr20)
                0 -1 -1   -4  -2  -2
                0 -1  0   -2  -4   2
                0  0 -1   -2   2  -4

                0 -1  0   -2  -4   2
                0 -1  0   -2  -4   2
                1  1  1    3   3   3   a,c not both null
               -1  0 -1   -1   1  -5
               -1  0  0    1  -1  -1
                0  0 -1   -2   2  -4
```

```
(-1,-1,-1)     -1  0  0    1  -1  -1
                1  0  0   -1   1   1
               -1 -1 -1   -3  -3  -3   This is (Tr21)
                0  1  1    4   2   2
                0 -1  0   -2  -4   2
                0  0 -1   -2   2  -4

                0  0 -1   -2   2  -4
                0  0  1    2  -2   4
               -1 -1 -1   -3  -3  -3   a',c not both null
                1  1  0    1   5  -1
               -1  0  0    1  -1  -1
                0 -1  0   -2  -4   2

                                ***************

               -1  0  0    1  -1  -1
                1  0  0   -1   1   1
               -1 -1 -1   -3  -3  -3   a,c not both null
                0  1 -1    0   6  -6
                0 -1  0   -2  -4   2
                0  0  1    2  -2   4

                0 -1  0   -2  -4   2
                0  1  0    2   4  -2
               -1 -1 -1   -3  -3  -3         ditto
                1  0 -1   -3   3  -3
               -1  0  0    1  -1  -1
                0  0  1    2  -2   4

                0 -1  0   -2  -4   2
                0  1  0    2   4  -2
               -1 -1 -1   -3  -3  -3         ditto
               -1  0  1    3  -3   3
                0  0 -1   -2   2  -4
                1  0  0   -1   1   1

                                ***************

               -1  0  0    1  -1  -1
                1  0  0   -1   1   1
               -1 -1 -1   -3  -3  -3
                0 -1 -1   -4  -2  -2    This is (Tr22)
                0  1  0    2   4  -2
                0  0  1    2  -2   4

                0 -1  0   -2  -4   2
                0  1  0    2   4  -2
               -1 -1 -1   -3  -3  -3    a,c not both null
               -1  0 -1   -1   1  -5
                1  0  0   -1   1   1
                0  0  1    2  -2   4

                                ***************
```

```
(1,-1,-1)      -1  0  0    1  -1  -1
               -1  0  0    1  -1  -1
                1 -1 -1   -5  -1  -1    orthogonality fails
                0  1  1    4   2   2
                0 -1  0   -2  -4   2
                0  0 -1   -2   2  -4

                0 -1  0   -2  -4   2
                0 -1  0   -2  -4   2
               -1  1 -1    1   5  -7    a,c not both null
                1  0  1    1  -1   5
               -1  0  0    1  -1  -1
                0  0 -1   -2   2  -4

                          ***************

               -1  0  0    1  -1  -1
               -1  0  0    1  -1  -1
                1 -1 -1   -5  -1  -1    orthogonality fails
                0 -1  1    0  -6   6
                0  1  0    2   4  -2
                0  0 -1   -2   2  -4

                0 -1  0   -2  -4   2
                0 -1  0   -2  -4   2
               -1  1 -1    1   5  -7    a,c not both null
               -1  0  1    3  -3   3
                1  0  0   -1   1   1
                0  0 -1   -2   2  -4

                0 -1  0   -2  -4   2
                0 -1  0   -2  -4   2
               -1  1 -1    1   5  -7    a,c not both null
                1  0 -1   -3   3  -3
                0  0  1    2  -2   4
               -1  0  0    1  -1  -1

                          ***************

               -1  0  0    1  -1  -1
               -1  0  0    1  -1  -1
                1 -1 -1   -5  -1  -1    orthogonality fails
                0 -1 -1   -4  -2  -2
                0  1  0    2   4  -2
                0  0  1    2  -2   4

                0  0 -1   -2   2  -4
                0  0 -1   -2   2  -4
               -1 -1  1    1  -7   5    a,c not both null
               -1 -1  0   -1  -5   1
                1  0  0   -1   1   1
                0  1  0    2   4  -2

                          ***************
```

```
(-1,1,-1)       1  0  0   -1   1   1
               -1  0  0    1  -1  -1
               -1  1 -1    1   5  -7    orthogonality fails
                0 -1 -1   -4  -2  -2
                0 -1  0   -2  -4   2
                0  0  1    2  -2   4

                0  1  0    2   4  -2
                0 -1  0   -2  -4   2
                1 -1 -1   -5  -1  -1           ditto
               -1  0 -1   -1   1  -5
               -1  0  0    1  -1  -1
                0  0  1    2  -2   4

                0  1  0    2   4  -2
                0 -1  0   -2  -4   2
               -1 -1  1    1  -7   5    a, c not both null
               -1  0 -1   -1   1  -5
                0  0 -1   -2   2  -4
                1  0  0   -1   1   1

                                   ***************

                1  0  0   -1   1   1
               -1  0  0    1  -1  -1
               -1  1 -1    1   5  -7    a,c not both null
                0 -1  1    0  -6   6
                0 -1  0   -2  -4   2
                0  0 -1   -2   2  -4

                0  1  0    2   4  -2
                0 -1  0   -2  -4   2
                1 -1 -1   -5  -1  -1    orthogonality fails
               -1  0  1    3  -3   3
               -1  0  0    1  -1  -1
                0  0 -1   -2   2  -4

                0  1  0    2   4  -2
                0 -1  0   -2  -4   2
               -1 -1  1    1  -7   5    a,c not both null
                1  0 -1   -3   3  -3
                0  0 -1   -2   2  -4
               -1  0  0    1  -1  -1

                                   ***************

(1,1,-1)       -1  0  0    1  -1  -1
               -1  0  0    1  -1  -1
                1  1 -1   -1   7  -5    orthogonality fails
                0 -1 -1   -4  -2  -2
                0 -1  0   -2  -4   2
                0  0  1    2  -2   4

                0 -1  0   -2  -4   2
                0 -1  0   -2  -4   2
                1  1 -1   -1   7  -5    a,c not both null
               -1  0 -1   -1   1  -5
```

```
                -1  0  0    1  -1  -1
                 0  0  1    2  -2   4

                 0 -1  0   -2  -4   2
                 0 -1  0   -2  -4   2
                -1  1  1    5   1   1    orthogonality fails
                -1  0 -1   -1   1  -5
                 0  0 -1   -2   2  -4
                 1  0  0   -1   1   1

                                  ***************

                -1  0  0    1  -1  -1
                -1  0  0    1  -1  -1
                 1  1 -1   -1   7  -5    a,c not both null
                 0 -1  1    0  -6   6
                 0 -1  0   -2  -4   2
                 0  0 -1   -2   2  -4

                 0 -1  0   -2  -4   2
                 0 -1  0   -2  -4   2
                 1  1 -1   -1   7  -5           ditto
                -1  0  1    3  -3   3
                -1  0  0    1  -1  -1
                 0  0 -1   -2   2  -4

                 0 -1  0   -2  -4   2
                 0 -1  0   -2  -4   2
                -1  1  1    5   1   1    orthogonality fails
                 1  0 -1   -3   3  -3
                 0  0 -1   -2   2  -4
                -1  0  0    1  -1  -1

                                  ***************

(-1,1,1)        -1  0  0    1  -1  -1
                 1  0  0   -1   1   1
                -1  1  1    5   1   1
                 0 -1 -1   -4  -2  -2    orthogonality fails
                 0 -1  0   -2  -4   2
                 0  0 -1   -2   2  -4

                 0 -1  0   -2  -4   2
                 0  1  0    2   4  -2
                 1 -1  1   -1  -5   7    a,c not both null
                -1  0 -1   -1   1  -5
                -1  0  0    1  -1  -1
                 0  0 -1   -2   2  -4

                                  ***************
```

Next we consider triples with index sets

```
123
  345
12 4
```

```
220a)          1  1  0     equivalent to 159)
              -1 -1  0
              -1  0  1
               0 -1 -1
               0  0 -1
```

****

```
221)           1  1  0     equivalent to 43)
              -1 -1  0
              -1  0 -1
               0 -1 -1
               0  0  1
```

****

```
222)           1 -1  0     equivalent to 161)
              -1  1  0
              -1  0 -1
               0 -1 -1
               0  0  1
```

****

```
223)           1 -1  0     equivalent to 104)
              -1  1  0
              -1  0 -1
               0 -1  1
               0  0 -1
```

****

```
224)         Triangle

               1  -1   0    -3  -3   3
              -1  -1   0    -1  -5   1
              -1   0   0     3  -3   3     a,c not both null
               0   1  -1     0   6  -6
               0   0  -1    -2   2  -4

              -1   1   0     3   3  -3
              -1  -1   0    -1  -5   1
               0  -1   1     0  -6   6           ditto
               1   0  -1    -3   3  -3
               0   0  -1    -2   2  -4

               0   1  -1     0   6  -6
               0  -1  -1    -4  -2  -2
               1  -1   0    -3  -3   3     orthogonality fails
              -1   0   1     3  -3   3
              -1   0   0     1  -1  -1
```

****

```
225)          1  -1   0        equivalent to 125)
             -1  -1   0
             -1   0  -1
              0   1   1
              0   0  -1
```

****

```
226)          1  -1   0        equivalent to 129)
             -1  -1   0
             -1   0  -1
              0   1  -1
              0   0   1
```

****

227)-230) are obviously equivalent to the above configurations

```
231)         -1  -1   0        equivalent to 166)
             -1  -1   0
              1   0   1
              0   1  -1
              0   0  -1
```

****

```
231a)        -1  -1   0        equivalent to 42)
             -1  -1   0
              1   0  -1
              0   1  -1
              0   0   1
```

****

Next we consider triples with index sets

```
                                   123
                                     345
                                    234
```

```
232)          1  0  0   now   1 is in the 2-plane also.
             -1 -1  0         0
             -1 -1 -1        -1
              0  1  1         0
              0  0 -1        -1
```

This is parallelogram (P12).

```
              1  0  0   -1  1  1
             -1 -1  0   -1 -5  1
             -1 -1 -1   -3 -3 -3     orthogonality fails
              0  1  1    4  2  2
              0  0 -1   -2  2 -4
```

```
 0  1  0    2  4 -2
-1 -1  0   -1 -5  1
-1 -1 -1   -3 -3 -3     a,c not both null
 1  0  1    1 -1  5
 0  0 -1   -2  2 -4

 0  1  0    2  4 -2
 0 -1 -1   -4 -2 -2
-1 -1 -1   -3 -3 -3     orthogonality fails
 1  0  1    1 -1  5
-1  0  0    1 -1 -1
```

****

```
233)          1  0  0   now  -1 is in the 2-plane but
             -1 -1  0         0
             -1 -1  1         1
              0  1 -1         0
              0  0 -1        -1
```

the parallelogram and its subtriangles are not faces.

****

```
234)          1  0  0   now  -1 is on the 2-plane
             -1 -1  0         0
             -1 -1 -1        -1
              0  1 -1         0
              0  0  1         1
```

This is parallelogram (P13).

```
 1  0  0   -1  1  1
-1 -1  0   -1 -5  1
-1 -1 -1   -3 -3 -3      a,c not both null
 0  1 -1    0  6 -6
 0  0  1    2 -2  4

 0  1  0    2  4 -2
-1 -1  0   -1 -5  1
-1 -1 -1   -3 -3 -3             ditto
 1  0 -1   -3  3 -3
 0  0  1    2 -2  4

 0  1  0    2  4 -2
 0 -1 -1   -4 -2 -2
-1 -1 -1   -3 -3 -3      a', c not both null
-1  0  1    3 -3  3
 1  0  0   -1  1  1

 1  0 -1   -3  3 -3
-1  0  0    1 -1 -1
-1 -1 -1   -3 -3 -3      a', c not both null
 0 -1  0   -2 -4  2
 0  1  1    4  2  2
```

```
         0  1 -1    0  6 -6
         0 -1  0   -2 -4  2
        -1 -1 -1   -3 -3 -3      a,c not both null
        -1  0  0    1 -1 -1
         1  0  1    1 -1  5

        -1  1  0    3  3 -3
         0 -1  0   -2 -4  2
        -1 -1 -1   -3 -3 -3            ditto
         0  0 -1   -2  2 -4
         1  0  1    1 -1  5
```

****

```
235)        Triangle

        -1  0  0    1 -1 -1
        -1 -1  0   -1 -5  1
         1 -1 -1   -5 -1 -1     orthogonality fails
         0  1 -1    0  6 -6
         0  0  1    2 -2  4

         0 -1  0   -2 -4  2
        -1 -1  0   -1 -5  1
        -1  1 -1    1  5 -7     a,c not both null
         1  0 -1   -3  3 -3
         0  0  1    2 -2  4

         0 -1  0   -2 -4  2
         0 -1 -1   -4 -2 -2
        -1  1 -1    1  5 -7     orthogonality fails
        -1  0  1    3 -3  3
         1  0  0   -1  1  1
```

****

```
236)    -1  0  0   now  -1  is in the 2-plane
        -1 -1  0         0
         1 -1 -1         1
         0  1  1         0
         0  0 -1        -1
```

This is parallelogram P14.

```
        -1  0 -1   -1  1 -5
        -1 -1  0   -1 -5  1
         1 -1  1   -1 -5  7     a,c not both null
         0  1  0    2  4 -2
         0  0 -1   -2  2 -4

         0 -1 -1   -4 -2 -2
        -1 -1  0   -1 -5  1
        -1  1  1    5  1  1     orthogonality fails
         1  0  0   -1  1  1
         0  0 -1   -2  2 -4
```

```
             -1 -1  0   -1 -5  1
              0 -1 -1   -4 -2 -2
              1  1 -1   -1  7 -5           ditto
              0  0  1    2 -2  4
             -1  0  0    1 -1 -1

             -1  0  0    1 -1 -1
             -1 -1  0   -1 -5  1
              1 -1 -1   -5 -1 -1     orthogonality fails
              0  1  1    4  2  2
              0  0 -1   -2  2 -4

              0 -1  0   -2 -4  2
             -1 -1  0   -1 -5  1
             -1  1 -1    1  5 -7     a,c not both null
              1  0  1    1 -1  5
              0  0 -1   -2  2 -4

              0 -1  0   -2 -4  2
              0 -1 -1   -4 -2 -2
             -1  1 -1    1  5 -7     orthogonality fails
              1  0  1    1 -1  5
             -1  0  0    1 -1 -1

                                ***************

237)         -1  0  0   now   1  is in the 2-plane
             -1 -1  0         0
              1 -1  1        -1
              0  1 -1         0
              0  0 -1        -1

This gives a parallelogram, but it and its subtriangles are not faces.

                                ***************

238)        Triangle

             -1  0  0    1 -1 -1
              1 -1  0   -3 -3  3
             -1 -1 -1   -3 -3 -3     a,c not both null
              0  1 -1    0  6 -6
              0  0  1    2 -2  4

              0 -1  0   -2 -4  2
             -1  1  0    3  3 -3
             -1 -1 -1   -3 -3 -3           ditto
              1  0 -1   -3  3 -3
              0  0  1    2 -2  4

              0 -1  0   -2 -4  2
              0  1 -1    0  6 -6
             -1 -1 -1   -3 -3 -3           ditto
             -1  0  1    3 -3  3
              1  0  0   -1  1  1

                        **********************
```

```
239)      Triangle

             -1  0  0    1 -1 -1
              1 -1  0   -3 -3  3
             -1 -1  1    1 -7  5     a,c not both null
              0  1 -1    0  6 -6
              0  0 -1   -2  2 -4

              0 -1  0   -2 -4  2
             -1  1  0    3  3 -3
             -1 -1  1    1 -7  5          ditto
              1  0 -1   -3  3 -3
              0  0 -1   -2  2 -4

              0 -1  0   -2 -4  2
              0  1 -1    0  6 -6
              1 -1 -1   -5 -1 -1     a,a' not both null
             -1  0  1    3 -3  3
             -1  0  0    1 -1 -1

                                     ***************

240)         -1  0  0        equivalent to 234)
              1 -1  0
             -1 -1 -1
              0  1  1
              0  0 -1

                                     ***************

241)       Triangle

              1  0  0   -1  1  1
             -1 -1  0   -1 -5  1     orthogonality
             -1  1 -1    1  5 -7     implies d_2 = 1
              0 -1 -1   -4 -2 -2     so a is not null
              0  0  1    2 -2  4

              0  1  0    2  4 -2
             -1 -1  0   -1 -5  1
              1 -1 -1   -5 -1 -1         ditto
             -1  0 -1   -1  1 -5
              0  0  1    2 -2  4

                                     ***************

242)          1  0  0       equivalent to 236)
             -1 -1  0
             -1  1  1
              0 -1 -1
              0  0 -1

                                     ***************
```

```
243)          1  0  0      now  -1   is in 2-plane.
             -1 -1  0            0
             -1  1 -1            1
              0 -1  1            0
              0  0 -1           -1
```

This parallelogram and its subtriangles are not faces.

****

```
244)         -1  0  0      now   -1   is in the 2-plane
             -1 -1  0             0
              1  1  1             1
              0 -1 -1             0
              0  0 -1            -1
```

This is parallelogram (P15).

```
             -1  0  0       1 -1 -1
             -1 -1  0      -1 -5  1
              1  1  1       3  3  3    orthogonality fails
              0 -1 -1      -4 -2 -2
              0  0 -1      -2  2 -4

              0 -1  0      -2 -4  2
             -1 -1  0      -1 -5  1
              1  1  1       3  3  3    a,c not both null
             -1  0 -1      -1  1 -5
              0  0 -1      -2  2 -4
```

****

```
245)         -1  0  0       now    1   is in the 2-plane
             -1 -1  0              0
              1  1 -1             -1
              0 -1  1              0
              0  0 -1             -1
```

The parallelogram and its subtriangles are not faces.

****

```
246)       triangle

             -1  0  0
              1 -1  0
             -1  1 -1
              0 -1  1
              0  0 -1

             -1       0       0       1      -1      -1
              1      -1       0      -3      -3       3
             -1       1      -1       1       5      -7    a,c not both null
              0      -1       1       0      -6       6
              0       0      -1      -2       2      -4
```

```
 0     -1      0     -2     -4      2
-1      1      0      3      3     -3
 1     -1     -1     -5     -1     -1   orthogonality fails
-1      0      1      3     -3      3
 0      0     -1     -2      2     -4
```

```
**************************************************
**************************************************
```

Finally, we check which triangles can satisfy the conditions of Theorem 6.12(ii). Recall that, up to permutation of x'', x, x', we may choose x'' to be type I. We shall take $x''_1 = -1$. Now nullity of a,a' implies $x_1 = x'_1$.

```
 x''        x   x'           c    a    a'

-1          1   1            3   -1   -1         a,c not both null
 0         -2   0           -2   -2    2
 0          0  -2           -2    2   -2

           -2  -2           -3   -1   -1               ditto
            1   0            1    1   -1
            0   1            1   -1    1

            0   0            1   -1   -1               ditto
           -2  -2           -4    0    0
            1   0            1    1   -1
            0   1            1   -1    1

            0   0            1   -1   -1
            1   1            2    0    0               ditto
           -2   0           -2   -2    2
            0  -2           -2    2   -2

            0   0            1   -1   -1
           -2   1           -1   -3    3               ditto
            1  -2           -1    3   -3

            0   0            1   -1   -1
           -2   0           -2   -2    2               ditto
            1  -2           -1    3   -3
            0   1            1   -1    1

            0   0            1   -1   -1
           -2   0           -2   -2    2          This is (Tr23)
            1   0            1    1   -1
            0  -2           -2    2   -2
            0   1            1   -1    1

            1   1            3   -1   -1
           -1  -1           -2    0    0          a,c not both null
           -1   0           -1   -1    1
            0  -1           -1    1   -1
```

1	1	3	-1	-1	
-1	0	-1	-1	1	
-1	0	-1	-1	1	ditto
0	-1	-1	1	-1	
0	-1	-1	1	-1	

1	1	3	-1	-1	
-2	-1	-3	-1	1	ditto
0	-1	-1	1	-1	

1	1	3	-1	-1	
-2	0	-2	-2	2	ditto
0	-1	-1	1	-1	
0	-1	-1	1	-1	

0	0	1	-1	-1	
1	1	2	0	0	ditto
-2	-1	-3	-1	1	
0	-1	-1	1	-1	

0	0	1	-1	-1	
1	-1	0	2	-2	ditto
-2	1	-1	-3	3	
0	-1	-1	1	-1	

0	0	1	-1	-1	
1	-1	0	2	-2	
-2	-1	-3	-1	1	This is (Tr24)
0	-1	1	-1	1	

0	0	1	-1	-1	
1	1	2	0	0	
-2	0	-2	-2	2	a,c not both null
0	-1	-1	1	-1	
0	-1	-1	1	-1	

0	0	1	-1	-1	
1	-1	0	2	-2	
-2	0	-2	-2	2	ditto
0	-1	-1	1	-1	
0	1	1	-1	1	

0	0	1	-1	-1	
1	0	1	1	-1	
-2	-1	-3	-1	1	ditto
0	1	1	-1	1	
0	-1	-1	1	-1	

0	0	1	-1	-1	
1	0	1	1	-1	
-2	1	-1	-3	3	ditto
0	-1	-1	1	-1	
0	-1	-1	1	-1	

```
 0   0             1   -1   -1
 1   0             1    1   -1
-2   0            -2   -2    2            This is (Tr25)
 0   1             1   -1    1
 0  -1            -1    1   -1
 0  -1            -1    1   -1

 0   0            now  0  is present,
 1  -1                 0
-1   1                 0
-1  -1                -1

 so it's not a simple triangle and this can be ruled out


 0   0             1   -1   -1
 1   1             2    0    0
-1  -1            -2    0    0           a,c not both null
-1   0            -1   -1    1
 0  -1            -1    1   -1

 0   0             1   -1   -1
 1  -1             0    2   -2
-1   1             0   -2    2                ditto
-1   0            -1   -1    1
 0  -1            -1    1   -1

 0   0             1   -1   -1
 1  -1             0    2   -2           This is (Tr26)
-1  -1            -2    0    0
-1   0            -1   -1    1
 0   1             1   -1    1

 0   0             1   -1   -1
 1   0             1    1   -1
-1  -1            -2    0    0           a,c not both null
-1  -1            -2    0    0
 0   1             1   -1    1

 0   0             1   -1   -1
 1   1             2    0    0
-1   0            -1   -1    1                ditto
-1   0            -1   -1    1
 0  -1            -1    1   -1
 0  -1            -1    1   -1

 0   0             1   -1   -1
 1  -1             0    2   -2
-1   0            -1   -1    1                ditto
-1   0            -1   -1    1
 0  -1            -1    1   -1
 0   1             1   -1    1
```

0	0	1	-1	-1	
1	0	1	1	-1	
-1	-1	-2	0	0	a,c not both null
-1	0	-1	-1	1	
0	1	1	-1	1	
0	-1	-1	1	-1	

0	0	1	-1	-1	
1	0	1	1	-1	
-1	0	-1	-1	1	
-1	0	-1	-1	1	This is (Tr27)
0	1	1	-1	1	
0	-1	-1	1	-1	
0	-1	-1	1	-1	

We do not need to consider examples where x or x' is a type II with a nonzero entry in place 1, as then x'' = (-1,...) is not a vertex.

0	0	1	-1	-1	
-1	0	-1	-1	1	This is (Tr28)
0	-1	-1	1	-1	