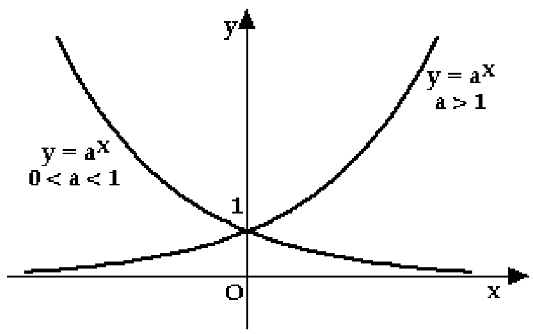
7. Exponential and Logarithmic Functions

This section contains review material on:

- Exponential functions and the natural exponential function
- Logarithmic functions and the natural logarithmic function

Exponential Functions. An exponential function is a function of the form $y = a^x$, where $a > 0$ and x is any real number. Although we can sometimes compute a power of a negative number, such as $(-4)^3$, the exponential function is defined only for positive bases. The domain of $y = a^x$ consists of all real numbers. Since $a^x > 0$ for all x (remember that $a > 0$!), it follows that the range of the exponential function $y = a^x$ consists of positive numbers only.

By plotting points, we obtain the graph of $y = a^x$.

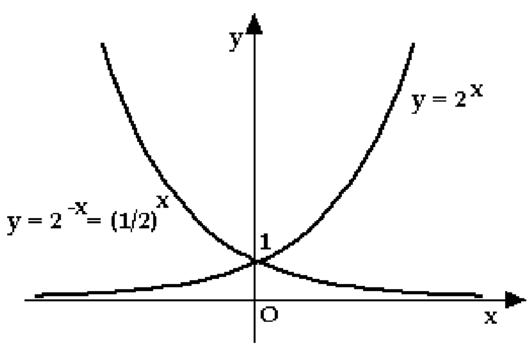


Since $a^0 = 1$, the graph of $y = a^x$ goes through the point $(0, 1)$ on the y-axis. If $a > 1$, the graph of $y = a^x$ is increasing. For $0 < a < 1$, it is decreasing. In either case, the x-axis is its horizontal asymptote.

Example 1. Sketch the graphs of $y = 2^x$ and $y = 2^{-x}$ in the same coordinate system.

Solution

Since $2^{-x} = (2^{-1})^x = (1/2)^x$, we are asked to plot the functions 2^x and $(1/2)^x$. By plotting points, we obtain the following picture.



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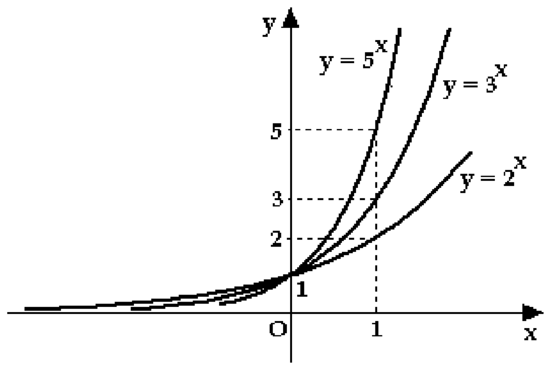
Example 2. Sketch the graphs of $y = 2^x$, $y = 3^x$ and $y = 5^x$ in the same coordinate system.

Solution

By computing the values of the given functions for different x, we see that as the basis a in $y = a^x$ increases, the graph increases faster and faster.



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Exercise 1. Sketch the graphs of the following functions.
(a) $y = 2^x + 4$ (b) $y = 2^{x-4}$ (c) $y = -2^x$ (d) $y = -2^{-x}$

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Exercise 2. Sketch the graphs of $y = 2^{-x}$, $y = 3^{-x}$ and $y = 4^{-x}$ in the same coordinate system.

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Although algebraic rules for working with exponential functions have been given already, we repeat them here for convenience.

$$a^0 = 1 \qquad a^1 = a \qquad a^x a^y = a^{x+y} \qquad (a^x)^y = a^{xy}$$
$$\frac{a^x}{a^y} = a^{x-y} \qquad \frac{1}{a^x} = a^{-x}$$

Example 3. Simplify the following expressions (i.e., reduce to a single exponential function).
(a) $4^{x+6} \cdot 8^{2-x}$ (b) $\frac{27^{2x-3}}{9^{x-4}}$ (c) $(2^x)^3 \cdot (4^{2-x})^4$.

Solution

(a) Using the above formulas, we get

$$\begin{aligned} 4^{x+6} \cdot 8^{2-x} &= (2^2)^{x+6} \cdot (2^3)^{2-x} = 2^{2(x+6)} \cdot 2^{3(2-x)} \\ &= 2^{2x+12} \cdot 2^{6-3x} = 2^{(2x+12)+(6-3x)} = 2^{-x+18}. \end{aligned}$$

(b) Similarly,

$$\frac{27^{2x-3}}{9^{x-4}} = \frac{(3^3)^{2x-3}}{(3^2)^{x-4}} = \frac{3^{6x-9}}{3^{2x-8}} = 3^{(6x-9)-(2x-8)} = 3^{4x-1}.$$

(c) Start by exponentiating the exponents:

$$(2^x)^3 \cdot (4^{2-x})^4 = 2^{3x} \cdot 4^{8-4x} = 2^{3x} \cdot (2^2)^{8-4x} = 2^{3x} \cdot 2^{16-8x} = 2^{-5x+16}. \qquad \blacksquare$$

Exercise 3. Simplify the following expressions (i.e., reduce to a single exponential function).

(a) $5^{x-2} \cdot 25^{3-x}$ (b) $3^{x-1} \cdot 9^{x-2} \cdot 27^{x-3}$ (c) $\dfrac{8^{x+4}}{16^{x-2}}$.

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Example 4. Solve each of the following equations for x.

(a) $4^x = 16^{2x-2}$ (b) $2^{x^3} = 0.25$ (c) $3^{2x} - 6 \cdot 3^x - 27 = 0$.

Solution

(a) Simplify so that both sides have the same basis:

$$4^x = 16^{2x-2}$$
$$4^x = (4^2)^{2x-2}$$
$$4^x = 4^{4x-4}$$

It follows that $4x - 4 = x$ and $x = 4/3$.

(b) Use the technique from (a):

$$2^{x^3} = 0.25 = \frac{1}{4} = \frac{1}{2^2} = 2^{-2}.$$

Thus, $x^3 = -2$ and so $x = \sqrt[3]{-2}$.

(c) The idea lies in the fact that $3^{2x} = (3^x)^2$; this implies that the given equation is a quadratic equation in 3^x. Let $y = 3^x$; then $3^{2x} - 6 \cdot 3^x - 27 = 0$ reads $y^2 - 6y - 27 = 0$. From

$$y^2 - 6y - 27 = (y+3)(y-9) = 0,$$

we conclude that $3^x = y = -3$ or $3^x = y = 9$.

Since $3^x > 0$, the equation $3^x = -3$ has no solutions. From $3^x = 9$, we get $x = 2$. Thus, the only solution is $x = 2$. ∎

Exercise 4. Solve each of the following equations for x.

(a) $0.5^{x^2} = 0.125$ (b) $3^x(3^x - 3) = 0$ (c) $2^{2x} - 5 \cdot 2^x + 4 = 0$.

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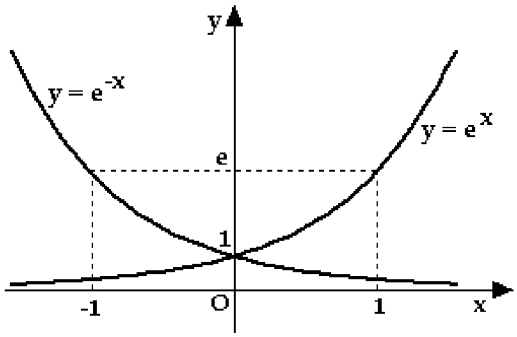
In the case when $a = e \approx 2.71828$, we obtain the so-called special exponential function $y = e^x$. This function is used in a number of applications, from population problems to compound interest and radioactive decay. The graphs of $y = e^x$ and $y = e^{-x} = 1/e^x$ are shown below.



Let us recall that (as any exponential function) the natural exponential function satisfies $e^0 = 1$ and $e^x > 0$ for all x.

Logarithms. The statement $a^m = n$ can also be written as $\log_a n = m$, where $\log_a$ is the logarithm to the base a. For example, $10^2 = 100$ is the same as $\log_{10} 100 = 2$. Similarly, $5^4 = 625$ can be restated as $\log_5 625 = 4$. The statement $\log_2 32 = 5$ is just another way of saying that $2^5 = 32$.

Substituting $m = \log_a n$ into $a^m = n$, we get $a^{\log_a n} = n$. In words, if we take a number (call in n), apply $\log_a$ to it and then exponentiate it (with the base a) we get our number back. Similarly, substituting $n = a^m$ into $m = \log_a n$, gives $\log_a a^m = m$. Thus, taking a number m, exponentiating it (with the base a) and then taking $\log_a$ does not change it.

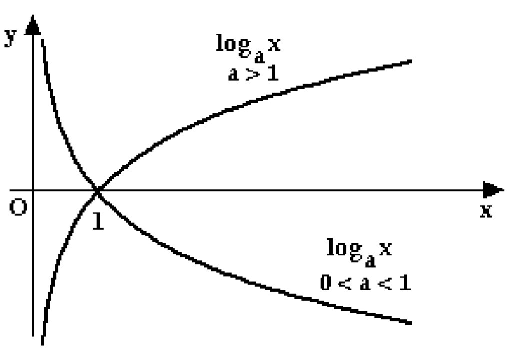
In other words, we say that exponentiating with the base a and applying logarithm to the base a are inverse of each other.

Note that from $n = a^m$ it follows that $n > 0$. Thus, $\log_a n$ is defined only for positive numbers n.

Logarithmic Functions. The logarithmic function $y = \log_a x$ is defined as the inverse function of the exponential function $y = a^x$. Consequently, when we apply the composition of the two functions (in any order) to a number x, we get it back:

$$a^{\log_a x} = x \qquad \text{and} \qquad \log_a a^x = x.$$

The domain of $\log_a x$ consists of positive numbers only. Its range is all of $\mathbb{R}$; see the graph below.



The graphs are the symmetric images of the graphs of $y = a^x$ with respect to the line $y = x$.

Since $a^0 = 1$, it follows that $\log_a 1 = 0$ (i.e., the value of $\log_a$ at 1 is 0). Thus, $\log_a x$ goes through the point $(1, 0)$ on the x-axis. If $a > 1$, $\log_a x$ is an increasing function; otherwise (if $0 < a < 1$), it is a decreasing function. In either case, the y-axis is its vertical asymptote.

Rules for logarithms

$$a^{\log_a x} = x \qquad \log_a a^x = x \qquad \log_a 1 = 0 \qquad \log_a a = 1$$

$$\log_a(xy) = \log_a x + \log_a y \qquad \log_a(x^n) = n \log_a x$$

$$\log_a(x/y) = \log_a x - \log_a y$$

Sometimes it might be useful to convert logarithms from one base to the other. The conversion formula is

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

The inverse function of the natural exponential function $y = e^x$ is called the natural logarithmic function, and is denoted by $\ln x$ (instead of $\log_e x$). Although we have already stated the properties of $y = \ln x$ when we talked about a general logarithmic function, we repeat it here.

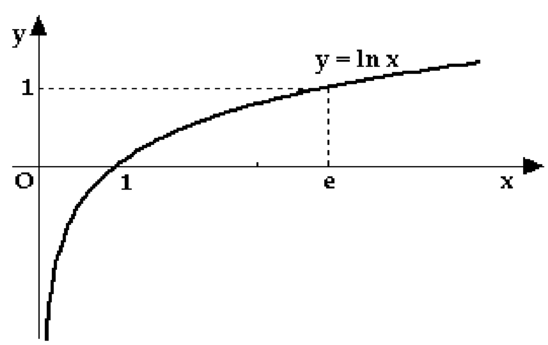
The domain of $\ln x$ is $(0, \infty)$. Its range consists of all real numbers.

By definition, e^x and $\ln x$ are inverse functions — thus, $e^{\ln x} = x$ (for all $x > 0$) and $\ln e^x = x$ (for all x). Moreover, $\ln 1 = 0$ and $\ln e = 1$ (the latter is true since $\ln e = \ln e^1 = 1$).

Natural logarithm can be used to simplify products, quotients and powers:

$$\ln(xy) = \ln x + \ln y \qquad \ln(x/y) = \ln x - \ln y \qquad \ln(x^n) = n \ln x.$$

The graph of $\ln x$ is given below (it is the symmetric image of $y = e^x$ with respect to the line $y = x$).



If needed, we can use conversion formulas

$$\ln x = \frac{\log_a x}{\log_a e} \qquad \text{and} \qquad \log_a x = \frac{\ln x}{\ln a}.$$

Example 5. Solve each of the following equations for x.

(a) $\log_2 x = 7$ (b) $\log_x 8 = 3$ (c) $\log_{16} 8 = x$ (d) $\log_2(\log_5 x) = 2$.

Solution

(a) Rewriting $\log_2 x = 7$ in the exponential form, we get $2^7 = x$; thus, $x = 64$. Alternatively, we could start with the equation $\log_2 x = 7$ and apply the exponential function 2^x to it, thus getting $2^{\log_2 x} = 2^7$; since $2^{\log_2 x} = x$, we get that $x = 2^7 = 64$.

(b) Rewriting $\log_x 8 = 3$ in the exponential form, we get $x^3 = 8$; thus, $x = 2$.

(c) Proceeding as in (a) or (b), we get $16^x = 8$. Thus $(2^4)^x = 2^3$, and $2^{4x} = 2^3$; it follows that $4x = 3$ and $x = 3/4$.

(d) Keep in mind the general principle: $\log_a B = C$ is equivalent to $B = a^C$. Applying this principle with $a = 2$, $B = \log_5 x$ and $C = 2$, we get $\log_5 x = 2^2 = 4$. Applying it once again, we get that $x = 5^4 = 625$. ∎

Exercise 5. Solve each of the following equations for x.

(a) $\log_x 4 = 1/2$ (b) $\log_3 x = 5$ (c) $\log_2 x^3 = \log_2(4x)$ (d) $16^{\log_4 x} = 4$.

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Example 6.

(a) Evaluate $e^{3\ln 2} \cdot e^{2\ln 3}$

(b) Express $2\ln 4 - \ln 8 - \ln 5$ as a single logarithm

(c) Solve $\ln(4x - 3) = 7$ for x

(d) Solve $\ln(\ln x) = 1$ for x

(e) Solve $\ln x + \ln(x + 7) = \ln 4 + \ln 2$ for x.

Solution

(a) We simplify exponents first and then use $e^{\ln x} = x$:

$$e^{3\ln 2} \cdot e^{2\ln 3} = e^{\ln 2^3} \cdot e^{\ln 3^2} = e^{\ln 8} \cdot e^{\ln 9} = 8 \cdot 9 = 72.$$

(b) Using $n \ln x = \ln x^n$ and $\ln x - \ln y = \ln(x/y)$, we get

$$\begin{aligned} 2\ln 4 - \ln 8 - \ln 5 &= \ln 4^2 - \ln 8 - \ln 5 = (\ln 16 - \ln 8) - \ln 5 \\ &= \ln(16/8) - \ln 5 = \ln 2 - \ln 5 = \ln(2/5). \end{aligned}$$

(c) Applying the inverse function e^x to both sides, we get

$$\begin{aligned} \ln(4x - 3) &= 7 \\ e^{\ln(4x-3)} &= e^7 \\ 4x - 3 &= e^7 \\ x &= \frac{e^7 + 3}{4}. \end{aligned}$$

(d) We repeat twice what we did in (c):

$$\begin{aligned} \ln(\ln x) &= 1 \\ e^{\ln(\ln x)} &= e^1 \\ \ln x &= e \\ e^{\ln x} &= e^e \\ x &= e^e. \end{aligned}$$

(e) Combining the terms on both sides we get

$$\begin{aligned} \ln x + \ln(x+7) &= \ln 4 + \ln 2 \\ \ln x(x+7) &= \ln 8 \\ x(x+7) &= 8 \\ x^2 + 7x - 8 &= 0 \\ (x+8)(x-1) &= 0. \end{aligned}$$

Thus, $x = -8$ and $x = 1$. The value $x = 1$ is a solution, since both terms on the right side of the given equation are defined. That is not true for $x = -8$, and so $x = 1$ is the only solution. ∎

Exercise 6.

(a) Evaluate $e^{\ln 4 + \ln 5}$

(b) Express $4 \ln 2 + \ln 3 + 2$ as a single logarithm

(c) Solve $e^{3x-2} = 4$ for x

(d) Solve $\ln(x^2 + x - 1) = 0$ for x.

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Additional exercises.

Exercise 7.

(a) Simplify $10 \cdot 100^2 \cdot 1000^4$ by reducing to a single exponential function

(b) Reduce $3^7 + 6 \cdot 3^6$ to a single term

(c) Reduce $9 \cdot 27^3 + 2 \cdot 3^{11}$ to a single term

(d) Simplify $\dfrac{36^{n+3}}{6^{2n+5}}$ by reducing to a single exponential function.

Exercise 8. Without a calculator, evaluate the following expressions.

(a) $\dfrac{(0.5 \cdot 10)^{-3}}{16 \cdot 0.1^4}$

(b) $0.2^{-4} \cdot 16$

(c) $-32 \cdot \left(\frac{1}{2}\right)^4$.

Exercise 9. Without a calculator, find numeric values of the following expressions.

(a) $e^{(1/2)\ln 8}$ (b) $10^{\log_{10} 5}$ (c) $\log_{10} 100000$.

Exercise 10. Without a calculator, find numeric values of the following expressions.

(a) $\log_3(1/9)$ (b) $\ln\left(e^{\ln(e^2)}\right)$ (c) $e^{-\ln 23}$.

Exercise 11. Solve the following equations.

(a) $0.1^x = 100$ (b) $(1/4)^x = 2$ (c) $0.25^x = 16$.

Exercise 12. Solve the following equations.

(a) $0.1^{x+2} = 100^{1/3}$ (b) $e^{2x} + 2e^x - 8 = 0$.