5. Functions

This section contains review material on:

- Definition of a function, domain, range
- Graph of a function
- Important graphs
- Creating new functions from old

Function. A function f is a rule that assigns, to each real number x in a set A a unique real number $f(x)$ in a set B. The set A is called the domain of f, and the set of all values $f(x)$ for all x in A is called the range of f.

We use the notation D for the domain and R for the range.

If a domain of f is not specified explicitly, and f is given by a formula (as in the next example), it is assumed that the domain of f consists of all real numbers for which the formula for f makes sense (i.e., defines a real number).

The value x is called the value of the independent variable, and $f(x)$ is called the corresponding value of the function f. The value $f(x)$ is also denoted by y; hence the usual notation $y = f(x)$.

Example 1. Find the domain and the range for the following functions.

(a) $f(x) = x^2$ (b) $g(x) = \dfrac{1}{x}$ (c) $h(x) = \sqrt{x}$.

Solution

(a) Since it is possible to compute the square of every real number, the domain of f consists of all real numbers. In symbols, $D = \mathbb{R}$. The square of any non-zero number is positive, and the square of zero is zero. Thus, the range of f consists of all y satisfying $y \geq 0$. In symbols, $R = [0, \infty)$.

(b) Division by zero is not allowed; thus, the domain of g consists of all real numbers x such that $x \neq 0$. Using set notation, we write $D = \{x \mid x \neq 0\}$.

We claim that the range is the set of all y such that $y \neq 0$. Since $1/x = 0$ has no solutions, it follows that no real number x can be mapped by g into $y = 0$; thus, $y = 0$ does not belong to the range of g. Pick any $y \neq 0$. From $1/x = y$ we get $x = 1/y$; thus, $g(1/y) = 1/(1/y) = y$, and so y is in the range of g (since it is obtained as the image of $1/y$ under g). So $R = \{y \mid y \neq 0\}$.

(c) The square root of a negative number is not defined; thus, $D = [0, \infty)$. By convention, $\sqrt{x} \geq 0$, and, consequently, $R = [0, \infty)$. ∎

Exercise 1. Find the domain and the range for the following functions.

(a) $f(x) = x^3$ (b) $g(x) = \dfrac{1}{x^2}$ (c) $h(x) = |x|$. ♣

Example 2. Find the domain for each of the following functions.

(a) $f(x) = \dfrac{1}{x+4}$ (b) $g(x) = \dfrac{x+2}{x^2-2x}$ (c) $h(x) = \sqrt{1-x^2}$.

Solution

(a) Since the division by zero is not allowed, it follows that all x in the domain of f must satisfy $x + 4 \neq 0$. Thus, $D = \{x \mid x \neq -4\}$.

(b) As in (a), we require that $x^2 - 2x \neq 0$. Since $x^2 - 2x = x(x-2) = 0$ implies that $x = 0$ or $x = 2$, we conclude that $D = \{x \mid x \neq 0 \text{ and } x \neq 2\}$.

(c) The square root function is not defined for negative numbers; thus, all x in the domain of h must satisfy $1 - x^2 \geq 0$, or $x^2 \leq 1$. Computing the square root of both sides, we get $\sqrt{x^2} \leq 1$, i.e., $|x| \leq 1$ (recall that $\sqrt{x^2} = |x|$). Thus, $-1 \leq x \leq 1$. Alternatively, we write $D = \{x \mid -1 \leq x \leq 1\}$.

We can avoid formalities in solving $x^2 \leq 1$ and proceed as follows. The square of a number between -1 and 1 is smaller than 1. Also, $1^1 = 1 \leq 1$ and $(-1)^2 = 1 \leq 1$. Thus, the solution of $x^2 \leq 1$ is the interval $[-1, 1]$. ∎

Exercise 2. Find the domain of the following functions.

(a) $f(x) = \dfrac{1}{\sqrt{x-2}}$ (b) $g(x) = \dfrac{x}{x^3-x}$ (c) $h(x) = \sqrt{1-\dfrac{1}{x}}$. ♣

Graph of a Function. The graph of a function $y = f(x)$ is the set of points $(x, f(x))$ for all values of x in the domain of f. In other words, the graph of f consists of points, and each point carries two pieces of data: the value of the independent variable x and the corresponding value $f(x)$ of the function f at x.

In general, the graphs of functions that we study in calculus are curves. An elementary way of sketching the graph is by plotting points.

The fact that a function assigns a unique value to each x in its domain means that its graph cannot contain two or more points that have the same x-coordinate. In other words, every vertical line that crosses the graph must cross it exactly once (this is known as the vertical line test).

We have already seen a few graphs of functions. For example, the graph of $y = mx + b$ is a line of slope m and y-intercept b; the graph of the function $y = ax^2 + bx + c$ is a parabola. An ellipse is not the graph of a function (the vertical line test does not work! — however, an ellipse is the graph of an equation).

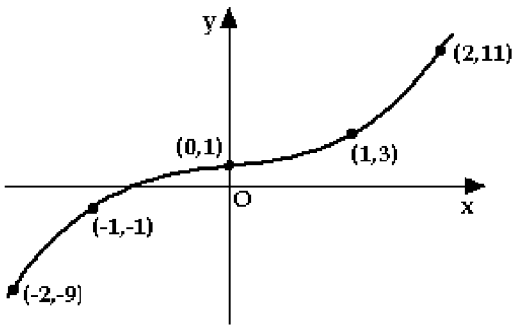
Example 3. Sketch the graph of the function $y = x^3 + x + 1$.

Solution

We choose (convenient) values for x and compute corresponding values for y. For example, when $x = 0$, we get that $y = 1$; thus, the point $(0, 1)$ belongs to the graph. In the same way, we can compute as many points as we wish. The table below shows some of them.

x	-2	-1	$-1/2$	0	$1/2$	1	2
$y = x^3 + x + 1$	-9	-1	$3/8$	1	$13/8$	3	11
point on the graph	$(-2,-9)$	$(-1,-1)$	$(-1/2, 3/8)$	$(0,1)$	$(1/2, 13/8)$	$(1,3)$	$(2,11)$

Now we use these points (the more points we have the better the picture will look like) to produce the following graph:



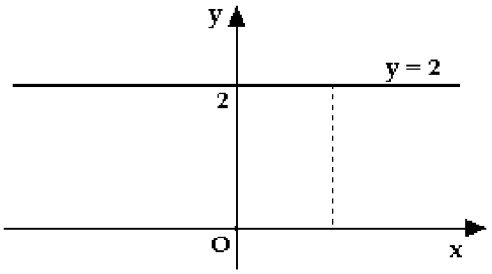
■

Exercise 3. Sketch the graph of the function $y = \sqrt{x^3 + 1}$ by plotting points.

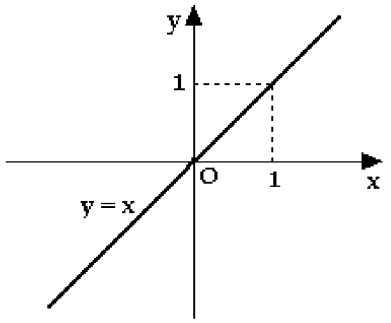
♣

Next, we present a list of important functions; for each function, we state its domain and range and show its graph.

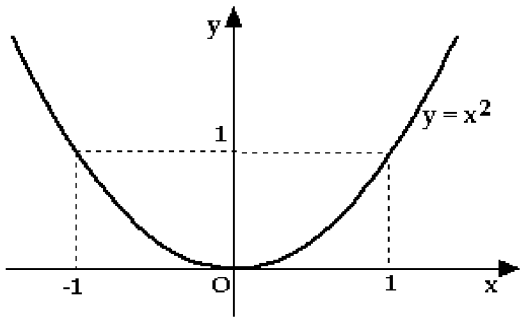
(1) $f(x) = c$ (c is a real number) is a constant function. It assigns the same number c to all real values x. Its domain consists of all real numbers. The range consists of a single point $\{c\}$. The graph below shows the constant function $f(x) = 2$.



(2) $f(x) = x$; $D = \mathbb{R}$, $R = \mathbb{R}$; the graph is a line of slope 1 through the origin.

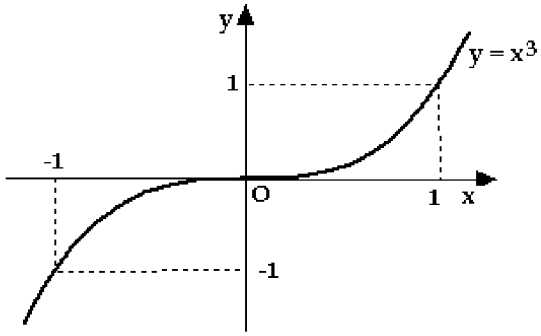


(3) $f(x) = x^2$; $D = \mathbb{R}$, $R = [0, \infty)$; the graph is a parabola with the vertex located at the origin, symmetric with respect to the y-axis.



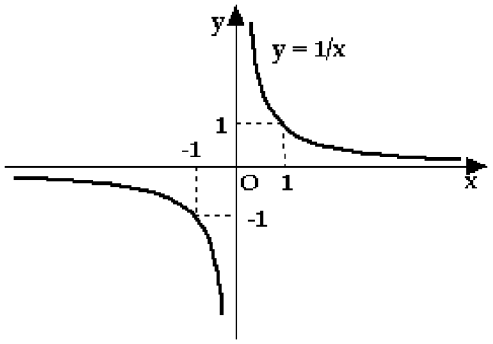
The graphs of x^n, where $n = 4, 6, 8, \ldots$, look similar to the graph of $f(x) = x^2$.

(4) $f(x) = x^3$; $D = \mathbb{R}$, $R = \mathbb{R}$; the graph is a cubic parabola; it is symmetric with respect to the origin.

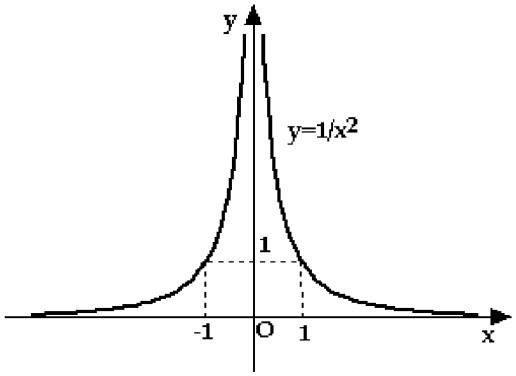


The graphs of x^n, where $n = 5, 7, 9, \ldots$, look similar to the graph of $f(x) = x^3$.

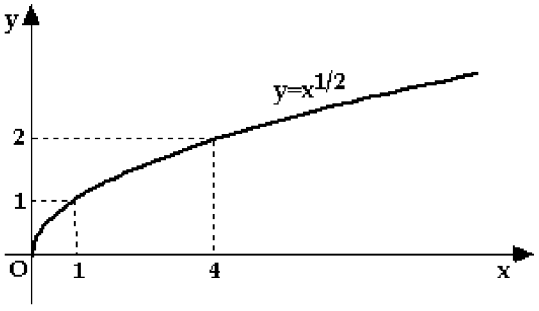
(5) $f(x) = 1/x$; $D = \{x \mid x \neq 0\}$, $R = \{y \mid y \neq 0\}$; the graph is a hyperbola, whose asymptotes are the x-axis and the y-axis. It is symmetric with respect to the origin.



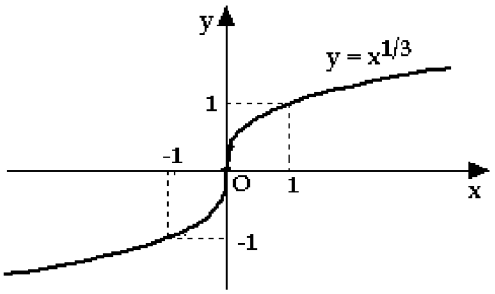
(6) $f(x) = 1/x^2$; $D = \{x \mid x \neq 0\}$, $R = \{y \mid y > 0\}$; the graph is a hyperbola, whose asymptotes are the x-axis and the y-axis. It is symmetric with respect to the y-axis.



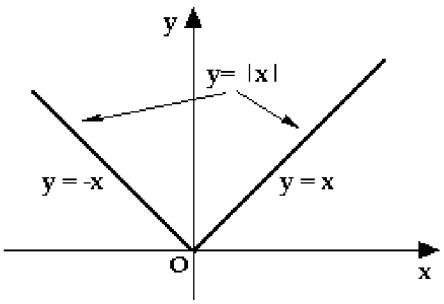
(7) $f(x) = \sqrt{x}$; $D = [0, \infty)$, $R = [0, \infty)$; the graph has a shape of a parabola.



(8) $f(x) = \sqrt[3]{x}$; $D = \mathbb{R}$, $R = \mathbb{R}$; the graph has a shape of a cubic parabola.



(9) $f(x) = |x|$; recall that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$; $D = \mathbb{R}$, $R = [0, \infty)$.



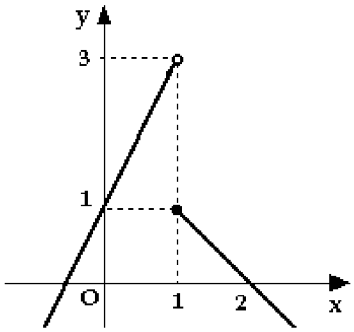
Example 4. Sketch the graph of the function

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ -x + 2 & \text{if } x \geq 1. \end{cases}$$

Solution

If $x < 1$, then $y = 2x + 1$; it is a line of slope 2 and the y-intercept 1. (Alternatively, we could compute two points that lie on it (say, $(0, 1)$ and $(1, 3)$) and join them with a straight line.) For $x \geq 1$, we get the line $y = -x + 2$ (of slope -1 and the y-intercept 2). The graph of f is given in the figure below. The filled dot in the graph indicates the fact that the value of f at $x = 1$ is $y = -x + 2 = -1 + 2 = 1$. The empty dot indicates that a point does not belong to the graph. (Note

that two filled dots, one above the other, would violate the vertical line test!).



Exercise 4. Sketch the graphs of the following functions.

(a) $f(x) = -4$ (b) $f(x) = -x^2$ (c) $f(x) = |x| + x$. ♣

New Functions From Old (Part I). Two functions can be added, subtracted, multiplied and divided (as long as the one in the denominator is not zero) to form new functions. For example, combining constant functions and the positive integer powers of x using addition, subtraction and multiplication, we obtain polynomials. In general, a polynomial of degree n is of the form $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, a_2, \ldots, a_n$ are real numbers. A quotient $p(x)/q(x)$ of two polynomials $p(x)$ and $q(x)$ (provided that $q(x) \neq 0$) is called a rational function. A function that can be obtained from polynomials using elementary algebraic operations and by taking roots is called an algebraic function. For example,

$$\sqrt{x}, \qquad \frac{x^4 + \sqrt[3]{x-1}}{x^2 + \sqrt{x}}, \qquad \frac{x^2 - 1}{x^6 + x - 1}, \qquad \sqrt[5]{\frac{1-\sqrt{x}}{x}}$$

are algebraic functions. The functions that are not algebraic are called transcendental. Trigonometric, logarithmic and exponential functions are examples of transcendental functions. We will study such functions later.

Another common way of obtaining new functions from old is by composing functions.

Composition of functions f and g is the function $g \circ f$, defined by
$(g \circ f)(x) = g(f(x))$

In words, we take an x and apply f to it, thus getting $f(x)$. Then we apply g to $f(x)$, to get the value $g(f(x))$ of the composition $g \circ f$ at x.

It is important to pay attention to the order in which the functions are composed. In general, $g \circ f \neq f \circ g$, as the following example shows.

Example 5. Let $f(x) = 1/x$ and $g(x) = x + 4$. Compute $g \circ f$ and $f \circ g$.

Solution

By definition,

$$(f \circ g)(x) = f(g(x)) = f(x + 4) = \frac{1}{x + 4}.$$

On the other hand,

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 4.$$

Clearly, $g \circ f \neq f \circ g$. ∎

The composition of three (or more) functions is computed analogously.

Example 6. Let $f(x) = \sqrt{x}$, $g(x) = 1 - \sqrt{x}$ and $h(x) = x^2 + 1$. Compute

(a) $f \circ g$ (b) $g \circ f \circ h$ (c) $f \circ h \circ f$.

Solution

(a) By definition,

$$(f \circ g)(x) = f(g(x)) = f(1 - \sqrt{x}) = \sqrt{1 - \sqrt{x}}.$$

(b) As in the case of a composition of two functions,

$$(g \circ f \circ h)(x) = g(f(h(x))) = g(f(x^2 + 1)) = g(\sqrt{x^2 + 1}) = 1 - \sqrt{\sqrt{x^2 + 1}} = 1 - \sqrt[4]{x^2 + 1}.$$

(c) Similarly,

$$(f \circ h \circ f)(x) = f(h(f(x))) = f(h(\sqrt{x})) = f((\sqrt{x})^2 + 1) = f(x + 1) = \sqrt{x + 1}.$$ ∎

Exercise 5. Let $f(x) = 3x + 4$ and $g(x) = x/2$. Compute

(a) $f \circ g$ (b) $f \circ f$ (c) $g \circ f$.

♣

Exercise 6. Let $f(x) = x^5$ and $g(x) = x^2 + x + 1$. Compute

(a) $f \circ g$ (b) $f \circ f$ (c) $g \circ f$.

♣

New Functions From Old (Part II). By applying certain operations to the graph of a function $y = f(x)$ we obtain new functions.

Assume that $c > 0$ is a constant. The function $y = f(x) + c$ is obtained by adding c to every value of $f(x)$; geometrically, it means that the points on the graph of $y = f(x) + c$ are obtained by moving points on the graph of $y = f(x)$ up for c units. Similarly, the graph of $y = f(x) - c$ is obtained by moving the graph of $f(x)$ c units down.

If $F(x) = f(x + c)$, then the value of F at x is the same as the value of f at $x + c$ (which is c units to the right of x); Thus, the graph of $F(x) = f(x + c)$ is obtained from the graph of $f(x)$ by moving it to the left for c units. The case of $F(x) = f(x - c)$ is argued analogously.

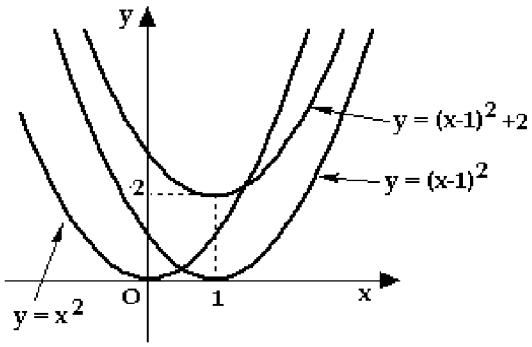
Let $c > 0$; to obtain the graph of		
$y = f(x) + c$,	move the graph of $y = f(x)$	c units up
$y = f(x) - c$,	move the graph of $y = f(x)$	c units down
$y = f(x + c)$,	move the graph of $y = f(x)$	c units to the left
$y = f(x - c)$,	move the graph of $y = f(x)$	c units to the right

Example 7. Sketch the graphs of the following functions.

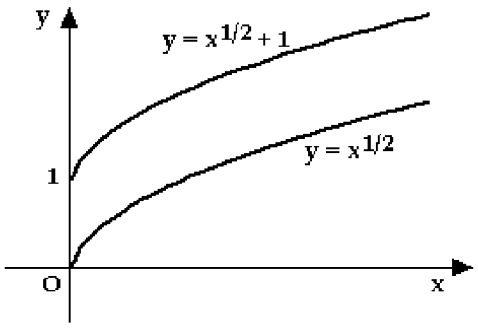
(a) $y = (x - 1)^2 + 2$ (b) $y = \sqrt{x} + 1$ (c) $y = \dfrac{1}{x - 4}$.

Solution

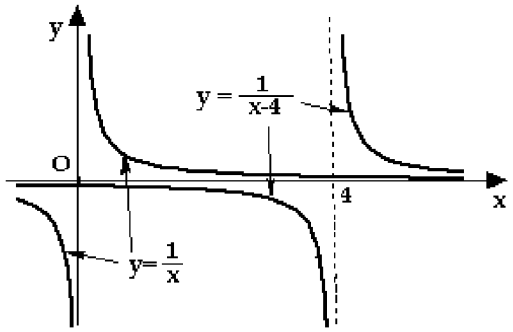
(a) The given graph is obtained from $y = x^2$ by replacing x by $x - 1$ and then by adding 2 to it; thus, we have to move the graph of $y = x^2$ one unit to the right and 2 units up.



(b) The graph of $y = \sqrt{x} + 1$ is obtained from the graph of $y = \sqrt{x}$ by moving it one unit up.



(c) The given graph is obtained from $y = 1/x$ by replacing x by $x - 4$; thus, all we need to do is to move the graph of $y = 1/x$ four units to the right.



∎

Exercise 7. Sketch the graphs of the following functions.

(a) $y = (x + 2)^2 + 3$ (b) $y = \sqrt{x - 4} - 2$ (c) $y = \dfrac{1}{(x - 1)^2} + 2.$

♣

A few more ways of constructing new graphs from old are given in the following table.

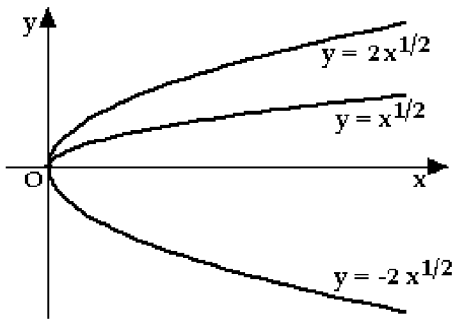
To obtain the graph of
$y = cf(x)$ $(c > 1)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = cf(x)$ $(0 < c < 1)$, compress the graph of $y = f(x)$ vertically by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ with respect to the x-axis

Example 8. Sketch the graphs of the following functions.

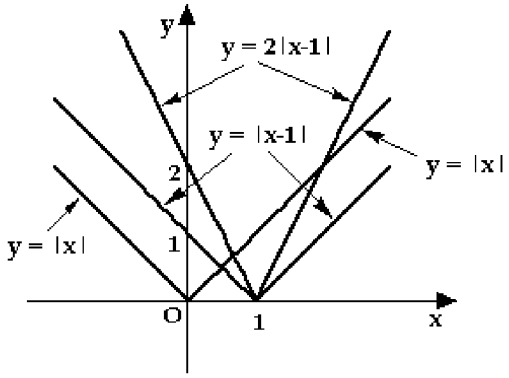
(a) $y = -2\sqrt{x}$ (b) $y = 2|x - 1|$.

Solution

(a) Take the graph of $y = \sqrt{x}$, expand it by a factor of 2 and then reflect with respect to the x-axis; see figure below.



(b) Start with the graph of $y = |x|$, move it 1 unit to the right and then expand it by a factor of 2.



■

Exercise 8. Sketch the graphs of the following functions.

(a) $y = -x^3/2$ (b) $y = 3\sqrt{x+3} + 1$.

♣

In the sections to follow, we will study other commonly used functions, such as trigonimetric, exponential and logarithmic functions.