More on WKB: Bender & Orszag form + higher order terms

We considered WKB approx. (based on multiple scales & transformed eqn) of

$$\ddot{x} + f(\varepsilon t)\, x = 0$$

BO consider the equation in the Boundary Layer form

$$\boxed{\varepsilon^2 y''(x) = Q(x)\, y(x)} \qquad (*)$$

if we let $\begin{cases} x = \varepsilon t \\ Q(x) = -f(\varepsilon t) \end{cases}$ we get

$\ddot{y}(t) + f(\varepsilon t)\, y(t) = 0$, which is the form we considered earlier. BO then assume ~~a series~~ a perturbation series in the form of a single exponential power series where both amplitude and phase are combined in the exponential, ie.

$$y(x) \sim \exp\left[\frac{1}{\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n S_n(x)\right] \qquad (**)$$

use dominant balance to

(they show boundary ~~layer~~ layer thickness $\delta \sim \varepsilon$)

They then simply sub $(**)$ in $(*)$ + compare powers of ε to get

"eikonal" $S_0'^2 = Q$ $\qquad S_0(x) = \pm \int^x [Q(x)]^{1/2} dt$

"transport" $2S_0'S_1' + S_0'' = 0$ $\qquad S_1(x) = -\frac{1}{4} \log Q(x)$ (equivalent to our t_0!)

$$S_2 = \pm \int^x \frac{Q''}{8Q^{3/2}} - 5\frac{(Q')^2}{32\,Q^{5/2}}\,dt$$

$$S_3 = -\frac{Q''}{16Q^2} + \frac{5Q'^2}{64Q^3}$$

etc. up to S_5. These results ~~can be transformed~~ are equivalent to our ~~per~~ previous results.

When is WKB valid?

For WKB to be valid on an interval, need $\sum \varepsilon^{n-1} S_n(x)$ to be an ~~uniformly~~ asymptotic series in ε as $\varepsilon \to 0$ uniformly $\forall x$ on the interval.

$\to$ a necessary condition ~~is~~ for asymptoticness is

①
$$\begin{cases} S_1(x) \ll \frac{1}{\varepsilon} S_0(x) & \varepsilon \to 0 \\ \varepsilon S_2(x) \ll S_1(x) & \varepsilon \to 0 \\ \vdots & \\ \varepsilon^n S_{n+1}(x) \ll \varepsilon^{n-1} S_n(x) & \varepsilon \to 0 \\ \text{or} \quad \dfrac{\varepsilon S_{n+1}(x)}{S_n(x)} \ll 1 & \text{uniformly in } x \text{ for } n = 0, \dots, N \end{cases}$$

But, since ~~are~~ $S_n(x)$ appear in the exponent

$$y(x) \sim \exp\left[\frac{1}{\varepsilon}\sum_{n=0}^{N} \varepsilon^n S_n(x)\right]$$

above conditions are not sufficient to ensure a good approx. to $y(x)$, also require that $\varepsilon^N S_{N+1}(x) \ll 1$ as $\varepsilon \to 0$. In this case we have $\exp\left[\varepsilon^N S_{N+1}(x)\right] = 1 + O\left[\varepsilon^N S_{N+1}(x)\right]$

If conditions ① hold then truncating $\sum_{n=0}^{N} \varepsilon^{n-1} S_n(x)$ before the smallest term $\varepsilon^N S_{N+1}(x)$ gives an asymptotic approx. to $\boxed{\ln y(x)}$ (not $y(x)$) with uniformly small error $\forall x$ in interval of interest ... but we are interested in $y(x)$, not $\ln y(x)$!

use condition ① →

$$-\ln y(x) + \frac{1}{\varepsilon} \sum_{n=0}^{N} \varepsilon^n S_n(x) \sim -\varepsilon^N S_{N+1}(x)$$

take exp

$$\rightarrow \frac{1}{y(x)} \exp\left(\frac{1}{\varepsilon} \sum_{n=0}^{N} \varepsilon^n S_n(x)\right) \sim \exp(-\varepsilon^N S_{N+1}(x))$$

$$\sim 1 - \varepsilon^N S_{N+1}(x) \quad \text{if} \quad \boxed{\varepsilon^N S_{N+1}(x) \ll 1} \text{ ②}$$

$$\rightarrow \frac{y(x) - \exp\left(\frac{1}{\varepsilon} \sum_{n=0}^{N} \varepsilon^n S_n(x)\right)}{y(x)} \sim \varepsilon^N S_{N+1}(x) \ll 1$$

as $\varepsilon \to 0$

we need both conditions ① + ② for asymptoticness of WKB approx.

Note that often $\boxed{\varepsilon = 1}$ problems are well-approximated by leading order (i.e. S_0, S_1) WKB.

⚠ This approach obscures the time-scale structure of problem, but is easier to derive.

EX. WKB with $\varepsilon = 1$: $y''(x) = \left(\frac{\ln x}{x}\right)^2 y(x)$

$$\rightarrow Q(x) = x^{-2} (\ln x)^2.$$

using the formulas:

$$S_0(X) = \pm\int^X \sqrt{Q(t)}\,dt = \pm(\ln X)^2$$

$$S_1(X) = -\tfrac{1}{4}\log Q(X) = \tfrac{1}{2}\log X - \tfrac{1}{2}\log\log X$$

$$S_2(X) = \pm\int^X \left(\frac{Q''}{8Q^{3/2}} - \frac{5Q'^2}{32Q^{5/2}}\right)dt =$$

$$\pm\tfrac{1}{8}\log\log X \pm \frac{3}{16}\frac{1}{(\log X)^2}$$

check ~~validity~~: $(\varepsilon=1)$ since $\varepsilon=1$, $X\to\infty$ is relevant asymptotic limit

① $S_2 \ll S_1 \ll S_0$ as $\boxed{X\to\infty}$ ∴ we have a uniformly valid asymptotic approximation

② However, $1 \ll S_2$ → need to take more than just S_0 & S_1.
But

$$S_3(X) = \frac{-Q''}{16Q^2} + \tfrac{5}{64}\frac{Q'^2}{Q^3} = \frac{3}{16}\frac{1}{(\log X)^4} - \frac{1}{16}\frac{1}{(\log X)^2}$$

& $S_3(X) \ll 1$ as $X\to\infty$

∴ leading-order approx to solution is given by:

$$y(X) \approx \exp[S_0(X) + S_1(X) + S_2(X)]$$

$$= c_\pm\, e^{\pm\frac{1}{2}(\log X)^2}\, X^{1/2}\,(\ln X)^{-1/2 \pm 1/8}$$

If $\varepsilon \neq 1$ ~~&~~ then leading-order using multiple-scales.

$y(t) \sim \exp[S_0/\varepsilon + S_1]$ for $X = O(1)$ (as we found before) since $X = \varepsilon t \to t = O(\varepsilon^{-1})$

But $\exp[S_0/\varepsilon + S_1]$ is a good leading-order approximation for x fixed as $\varepsilon \to 0$ since

$$\varepsilon S_2 \ll 1 .$$

If ε is fixed then $\exp[S_0/\varepsilon + S_1 + \varepsilon S_2]$ is the leading behavior as $x \to \infty$ (just like $\varepsilon = 1$) since εS_2 (error) $\to 0$ for any ε as $x \to \infty$!

$\boxed{\text{SHOW WKB1 Maple Ex.}}$

WKB method is very useful for estimating e-values & e-fcns of S-L problems.

Ex. e-value problem:

$$y''(x) = -\underbrace{E\,Q(x)}_{\text{weight function}}\, y(x), \quad Q(x) > 0, \quad y(0) = y(\pi) = 0$$

$\to$ normalize ~~e-val~~ e-fcns y_n s.t ~~$\langle y_n, y_n$~~

$$\langle y_n, y_n\rangle^{1/2} = \|y_n\| = \left[\int_0^\pi y_n^2 Q\, dx\right]^{1/2} = 1$$

$E_n \propto n^2$ as $n \to \infty$ $\therefore$ take $\varepsilon = 1/E_n$

& $\varepsilon y'' + Qy = 0$, $\varepsilon \to 0$

WKB + B.C. $y(0) = 0$

$$\to \quad y_n(x) \sim C_n Q^{-1/4}(x) \sin\left[\sqrt{E_n}\int_0^x \sqrt{Q(t)}\, dt\right], \quad E \to \infty \ (\propto n^2)$$

$$y(\pi) = 0 \ \to \ E_n = \left[\frac{n\pi}{\int_0^\pi \sqrt{Q(t)}\, dt}\right]^2, \quad n \to \infty \quad (\checkmark)$$

$$\& \ C_n^2 \sim \frac{2}{\int_0^\pi \sqrt{Q(t)}\, dt}, \quad n \to \infty \quad \text{(normalization)}$$

$\therefore$ e-fcns are : $y_n(x) \sim \left[\int_0^\pi \frac{\sqrt{Q(t)}}{2} dt\right]^{-1/2} Q^{-1/4}(x) \sin\left[\frac{n\pi \int_0^x \sqrt{Q(t)}dt}{\int_0^\pi \sqrt{Q(t)}dt}\right]$

<u>note</u>: if $Q(x) \equiv 1$ (eg. wave eqn on string)
then $y_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$ (<u>exact</u> sol^n!)

These results are <u>very</u> accurate, even if $n=1$ ($\sim 3\%$ error), when $Q(x) = (x+\pi)^4$.

The interval here is <u>finite</u> so we didn't have to worry about $x \to \infty$ (ie. $x = O(1)$).

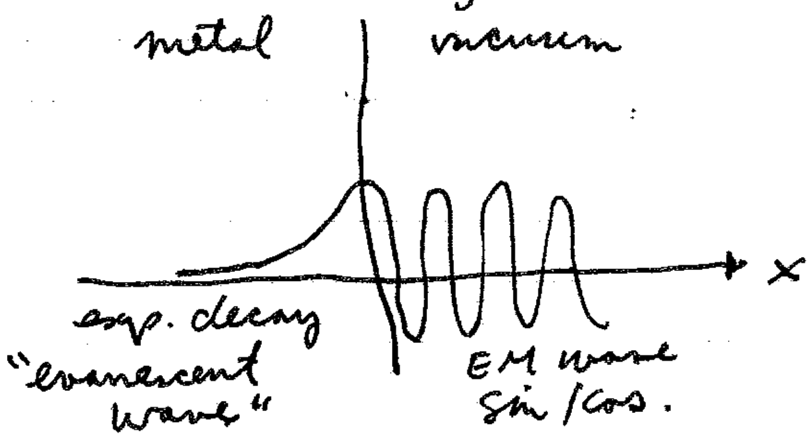
$S_2(x) \approx \frac{1}{3(x+\pi)^3}$ $\therefore$ we expect max error to be $\approx \frac{1}{3\pi^3} = 0.01 = 1\%$

in agreement with observation.

<u>Turning points</u>: what happens when frequency vanishes?

$\rightarrow$ WKB approx. gives an infinite amplitude; this is clearly not physical.

Consider the following situation in electromagnetism.



Physically, wave amplitude is not infinite at metal surface!

Also, from a purely mathematical point of view the theory of linear differential equations ensures that if frequency function $Q(\tau)$ or $f(\varepsilon t)$ is an analytic function then solution is regular.

typical example (from QM):

$$\varepsilon^2 y'' = (V(x) - E)\, y \qquad \text{"e-value problem"}$$

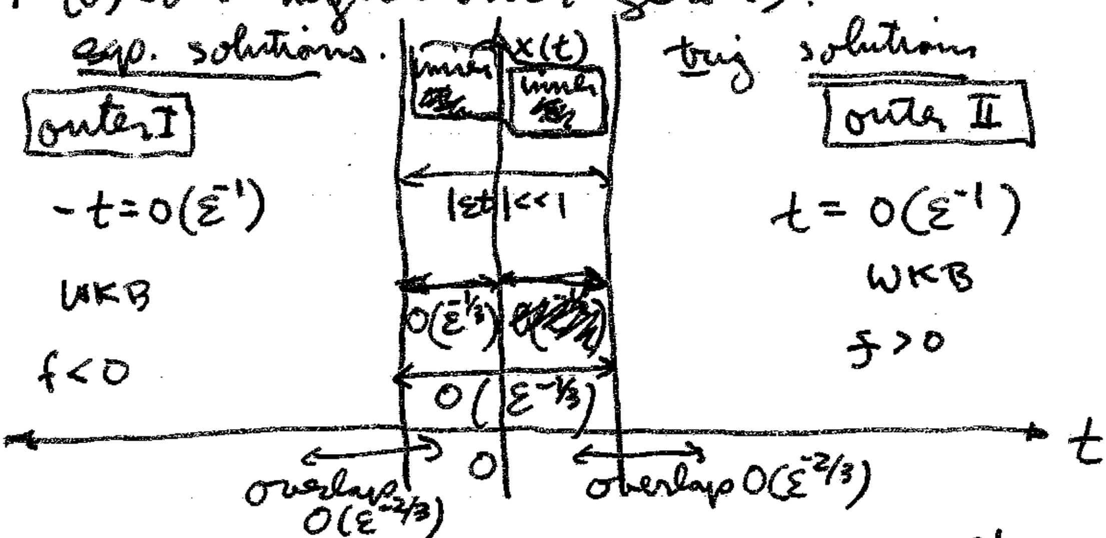
$V(x)$: pot. E: energy

$Q(x) = 0$ when $V(x) = E$ (classical particle is confined to $V(x) \leq E$)

Need some way of connecting solutions for $f > 0$ (oscillating) with $f < 0$ (exp. growing/decaying) → using matched asymptotic expansions.

Derivation of "turning pt." connection formulas.

wlog move $f=0$ pt. to origin, ie $f(0) = 0$ + assume $f'(0) > 0$ (in general, will have to consider $f'(0) < 0$ + higher-order zeroes).



$$\ddot{x} + f(\varepsilon t)\, x = 0 \quad ; \; f(0) = 0 \,, \; f'(0) > 0$$

OI & OII are given by WKB solutions for $f<0$ & $f>0$ respectively, $t=O(\varepsilon^{-1})$.

<u>Consider Inner</u>: here $|\varepsilon t| \ll 1$, $t<0$
assume $f(0) = 0 + \varepsilon t f'(0) + O(\varepsilon^2)$ (first-order zero)

$$\rightarrow \ddot{x} + \varepsilon t f'(0) x = 0$$

<u>rescale</u>: $\tau = -t\,(\varepsilon f'(0))^{1/3}$ to eliminate $\varepsilon f'(0)$ (ie. $t = O(\varepsilon^{-1/3})$) & B.L. $\delta = O(\varepsilon^{-1/3})$

$$\rightarrow x_{\tau\tau} - \tau x = 0$$

"Airy's eqn." } same eq. for inner 1 & 2

negative freq. $\tau>0$
positive " $\tau<0$

$$\rightarrow x = \alpha\, Ai(\tau) + \beta\, Bi(\tau), \quad \alpha \,\&\, \beta \text{ consts.}$$

(can evaluate $Ai(\tau)$ & $Bi(\tau)$ for $\tau \to \pm\infty$ using steepest descents)

Need to <u>match</u> Inner to Outer 1:
in matching range $\tau \to +\infty$, $t \nearrow 0$.

$$\text{inner} = \frac{1}{\tau^{1/4}\sqrt{\pi}}\left[\tfrac{1}{2}\alpha \exp\left(-\tfrac{2}{3}\tau^{3/2}\right) + \beta \exp\left(\tfrac{2}{3}\tau^{3/2}\right)\right]$$

(to leading order)

$$\text{outer 1} = \frac{1}{[-\varepsilon t f'(0)]^{1/4}}\left[A \exp(\varphi) + B \exp(-\varphi)\right]$$

$$\text{where } \varphi = -\tfrac{2}{3}\left[\varepsilon f'(0)\right]^{1/2}(-t)^{3/2} = -\tfrac{2}{3}\tau^{3/2}$$

to leading order.

(for $|t|$ small, $t<0$, $\varphi \approx \int_{-t}^{0} \sqrt{\varepsilon s f'(0)}\, ds = -\tfrac{2}{3}\left[\varepsilon f'(0)\right]^{1/2}(-t)^{3/2} = -\tfrac{2}{3}\tau^{3/2}$

To match we choose α & β in terms of A & B

$$\alpha = \frac{2\sqrt{\pi}}{[\varepsilon f'(0)]^{1/6}} A \qquad \beta = \frac{\sqrt{\pi}}{[\varepsilon f'(0)]^{1/6}} B$$

Now do same matching procedure for inner & outer 2: in overlap region $\tau \to -\infty$, $t \downarrow 0$

inner : $\frac{1}{(-\tau)^{1/4}\sqrt{\pi}}\left(\alpha \sin \Theta + \beta \cos \Theta\right)$

$$\Theta = 2/3(-\tau)^{3/2} + 1/4\pi$$

outer : $\frac{1}{[\varepsilon t f'(0)]^{1/4}}\left(a \cos\theta + b \sin\theta\right)$

$$\theta = 2/3\left[\varepsilon f'(0)\right]^{1/2} t^{3/2}$$

matching : $a = \frac{[\varepsilon f'(0)]^{1/6}}{\sqrt{2\pi}}(\alpha+\beta)$ & $b = \frac{[\varepsilon f'(0)]^{1/6}}{\sqrt{2\pi}}(\alpha-\beta)$

$\therefore$ we can connect A, B to a, b via α & β

$$\rightarrow \boxed{A = \frac{a+b}{2\sqrt{2}} \quad \& \quad B = \frac{a-b}{\sqrt{2}}}$$

usually the exponential solution is <u>decaying</u> as $t \to -\infty$

$\therefore$ $B = 0 \implies a \sim b \Rightarrow a = \sqrt{2}A$

This means the solution for $t > 0$ becomes:

$$x(t) = \sqrt{2}A\left[f(\varepsilon t)\right]^{-1/4}\left(\cos\theta + \sin\theta\right), \quad \theta = \int_0^t [f(\varepsilon t')]^{1/2} dt'$$