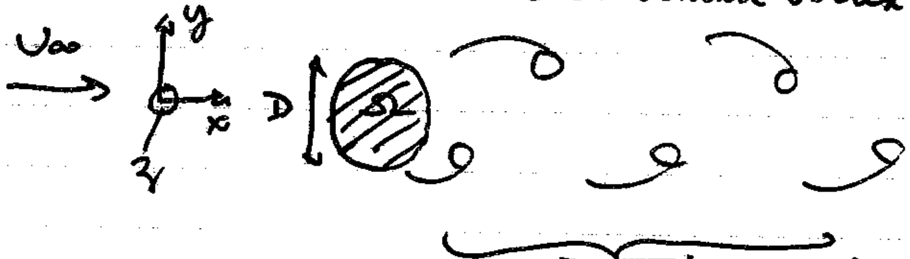
Base Flow (to be perturbed): 2-D time periodic vortex shedding flow past a circular cylinder.

"von Karman vortex street"



$\boxed{\text{vortices}}$, shedding frequency is Strouhal frequency

$St = \frac{fD}{U_\infty} \approx 0.195$ at $Re \approx 188$.

2-D time $\qquad$ of 2-D

N.B. Flow is exactly periodic with frequency given by Strouhal frequency ~~for~~ St(Re) for $\cancel{60} \; 45 \leq Re \leq 188$.

<u>Question</u>: at what Re does the flow become 3-D? Or, in other words, at what Re ~~is~~ does the flow become unstable to <u>spanwise</u> perturbations?

→ use <u>Floquet Stability Analysis</u>!

What is the base flow? A solution to the <u>2-D</u> Navier-Stokes equations.

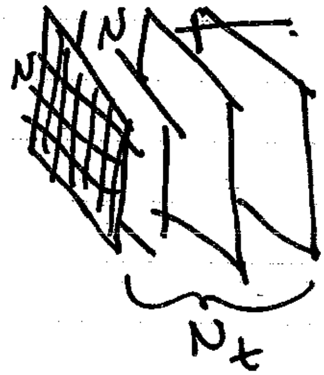
$$\partial_t u = -u \cdot \nabla u - \frac{1}{\rho}\nabla p + \frac{1}{Re}\Delta u \;, \quad \nabla \cdot u = 0$$

$$\bar{u} = u_\infty \;, \quad \text{B.C.} \quad u \to u_\infty \text{ as } |x| \to \infty. \qquad u|_{\partial\Omega} = 0 \text{ (no slip)}$$

solve <u>numerically</u> (eg. using a pseudo-spectral code) to obtain an exactly time periodic flow.

$$u(x, y, t_0 + nT) = u(x, y, t_0)$$

(!) this flow is known only at a finite number of spatial points, e.g. N^2, ~~etc~~ and at a finite number of times N_t.



Base Flow $u(\cancel{x,y,t+\omega t}) \equiv u$

$$u(x,y,t) = u(x,y,t+NT)$$

$\Rightarrow$ Base flow (unperturbed solution) is not known analytically!

Now, perturb with 3-D spanwise modes of the form

homogeneous in z $\rightarrow$ Fourier modes in z

$$\begin{cases} u'(x,y,z,t) = (\hat{u}\cos\beta z,\ \hat{v}\cos\beta z,\ \hat{w}\sin\beta z) \\ p'(x,y,z,t) = \hat{p}\cos\beta z \end{cases}$$

assuming $\dfrac{\|u'\|}{\|u\|} \ll 1$ we can solve the linearized N-S equations for u':

$$\partial_t u' = -\Big[\underbrace{(u'\cdot\nabla)u + (u\cdot\nabla)u'}_{\text{linearized advection term}}\Big] - \frac{1}{\rho}\nabla p' + \frac{1}{Re}\Delta u'$$

$$\nabla\cdot u' = 0$$

$$u'\big|_{\partial\Omega} = 0 \qquad u' \to 0 \text{ as } |x| \to \infty$$

or $$\boxed{\partial_t u' = -\mathbb{P}\Big[(u'\cdot\nabla)u_z + (u\cdot\nabla)u'\Big] - \frac{1}{\rho}\nabla p' + \frac{1}{Re}\Delta u'} \quad (*)$$

where $\mathbb{P}$ = divergence-free projection, e.g. normal to wavenumber $\vec{k}$: $\delta_{ij} - \dfrac{k_i k_j}{|k|^2}$

Formally this is a first order linear ODE in time:

$$\boxed{\frac{\partial u'}{\partial t} = L(u')}$$ for the linear operator L defined by ⊛

where L is T-periodic ~~in~~ in time & since base flow is.

∴ Look for solutions in terms of Floquet modes.

$u'(x,y,z,t) = \tilde{u}(x,y,z,t)\, e^{\sigma t}$ where $\tilde{u}$ are also T-periodic.

These are the eigenfunctions of L.

Floquet multipliers $\mu \triangleq e^{\sigma t} \rightarrow \begin{cases} |\mu|<1, \text{ stable} \\ |\mu|>1, \text{ unstable} \end{cases}$

or $\mathrm{Re}\,\sigma > 0$ unstable, $\mathrm{Re}\,\sigma < 0$ stable.

How to find Floquet modes?

① construct linear operator A which represents evolution of linear system over one period
(⇔ linearized Poincaré map associated with base flow)

: $\boxed{u'_{n+1} = A(u'_n)}$

e-values of A are Floquet multipliers of L, $e^{\sigma t}$.
e-fcns of A are Floquet modes of L at some time t_0.

② ∴ To find Floquet multipliers and Floquet modes we just have to find e-values & e-fcns of linear operator A.

Problems:

① We only know u at a fixed number of times N_t (eg. $N_t = 32$ per period) and we will need u at arbitrary times when solving linearized system.

→ use Fourier interpolation in time (spectral, exp., accuracy).

② Need a numerical method to find e-values & e-fcns of A (only need first few largest e-values in magnitude).

→ use an iterative method, such as Arnoldi, or Krylov subspace.

* since method is iterative need to solve linearized equations many times over one period. Convergence in 6-25 iterations of A.

Krylov subspace of size 20 gives good result, error $O(10^{-3})$ and first 20 e-fcns.

Results: (Barkley & Henderson 1996, JFM 322, 215-241)

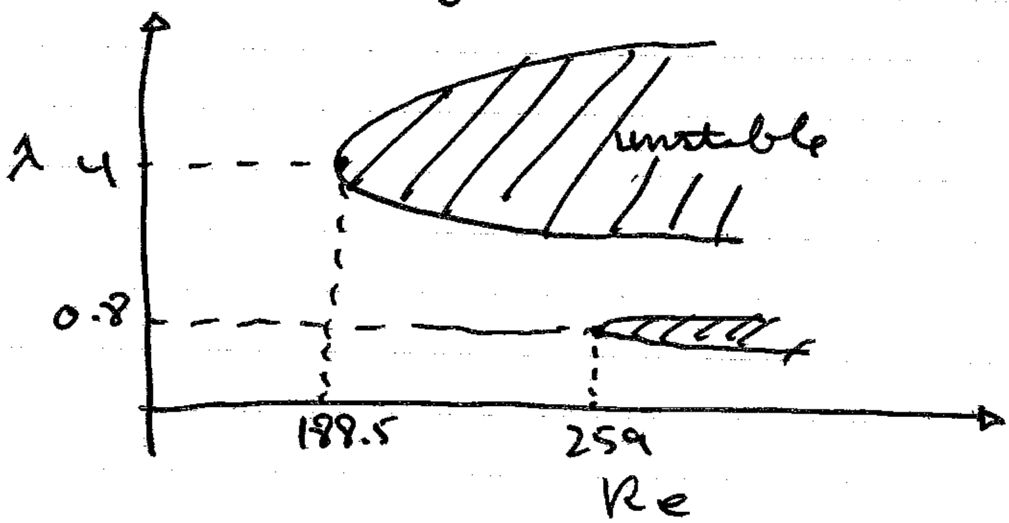
Try different Re & different spanwise β to determine Re_c at which any β is unstable.

$|\mu| = 1$ → $\boxed{Re_c = 188.5 \pm 1.0,\ \beta = 1.585}$ → $\boxed{\lambda = \frac{2\pi}{\beta} \approx 4D.}$

There is also a secondary instability mode with a shorter wavelength at

$$Re = 259 \pm 2, \quad \beta = 7.64 \pm 0.06, \quad \lambda = 0.822D.$$

summarize results as neutral stability curves (as in Mathieu equation):



Although this method is computationally intensive it is straightforward and feasible even when base flow is 2-D and only known numerically!

→ $\boxed{\text{show results for cylinder array}}$

Moral: can use Floquet stability analysis when analyzing stability of periodic phenomena to periodic perturbations, even when ~~base solution~~ /unperturbed solution is not known analytically. formof.

Summary of topics

1. Definition of an asymptotic series and asymptoticness; contrast between convergent & asymptotic series.

   How to use asymptotic series in approximations; how many terms to take
   Big "O" & little "o", allowable functions $\delta_n(\varepsilon)$.

2. Asymptotic approximation of integrals
   - integration by parts
   - Laplace's method, Watson's Lemma
   - Stationary phase
   - Steepest descents (most general)
   - non-local contributions (both local & non-local contributions can appear at any order
     → relation to Boundary Layer Theory.

3. Perturbation series solutions
   - regular, singular, matching.

4. Boundary layer Theory
   - outer, inner, matching. → uniform approx.
   - logarithms $\varepsilon \ln \frac{1}{\varepsilon}$
   - how thick is B.L. → dominant balance, distinguished limit.
   - where is B.L. (eg. internal).
   - nested B.L. (eg. van der Pol for large damping).
   - comparison with numerical solutions & evaluation of error.

5. Multiple Scales

- type of problem (slow drift in amplitude / frequency)

- van der Pol oscillator for small damping.
- multiple scale method: introduce hierarchy of slow time scales (treated as new variables) & use new flexibility to eliminate resonant (secular) terms.

6. WKB approx. (special case of multiple scales)

$$\ddot{x} + f(\varepsilon t)x = 0$$

- derivation.
- Orszag form

- higher order terms & validity (how many terms to take for a given $f(\varepsilon t)$).
- application to e-value problems.

- turning points (i.e. where $f(\varepsilon t) = 0$).
  → B.L. structure.
- inhom. equations → Green's function method

7. Floquet Stability Analysis
- linear stability ~~anal~~ analysis of ODEs with periodic solution & periodic perturbations.
- use multiple scales.

EX. * Mathieu eqn.
  * numerical Floquet analysis.