MATHEMATICS 744

McMaster University Final Examination — Dr. N. Kevlahan
Posted: 15 December 2010 at 9:30am — Due: 17 December 2010 at 9:30am in HH-324

Instructions

1. Please show all work and calculations. All **four** questions should be attempted.
2. You may consult the textbooks (i.e. Bender & Orszag and Hinch) and the course notes, but no other references (including the internet) are allowed. The questions should not be discussed with anyone else.
3. The total number of marks is 100 (marks are indicated in the margin).

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1. **Asymptotic expansions**

[6] (a) Compare the advantages and disadvantages of convergent and divergent series for obtaining useful approximations.

[6] (b) How does one use a divergent series to provide an accurate approximation?

[8] (c) Prove that the Taylor series of a function about a point z_0, where it is analytic, is also an asymptotic expansion. You should comment on the difference between proving asymptoticness and proving convergence.

2. **Asymptotic approximation of integrals**

[6] (a) Use the method of stationary phase to evaluate the leading order term in the asymptotic approximation of the following integral,

$$I(x) = \int_0^1 \exp(ixt^3)\,\mathrm{d}t, \quad x \to \infty.$$

[4] (b) Explain why the method of stationary phase can, in general, give only the leading order asymptotic approximation to an integral.

[14] (c) Use the method of Steepest descents to evaluate the first *two* terms in the asymptotic approximation of the integral in part (a). Check that the leading order term is the same as the one you found in part (a). Include a sketch indicating the contours you use and explain the origin of the second-order term that was missing from the stationary phase approximation.

3. **Boundary layers**

[6] (a) Could a uniform asymptotic approximation to a boundary layer problem be constructed matching the inner and outers solutions at a single point? Why or why not?

[12] (b) Find the leading-order uniform approximation to the boundary-value problem

$$\epsilon y''(x) + \left(x^{1/2} - \frac{\epsilon}{2x}\right) y'(x) - y(x) = 0,$$

with boundary conditions $y(0) = 0$, $y(1) = e^2$. You may assume that the boundary layer is located at $x = 0$. What is the thickness of the boundary layer?

[8] (c) Use the `matlab` function `bvp4c` to plot the 'exact' numerical solution and leading order asymptotic approximation when $\epsilon = 0.01$. [Hint: you will need to use the 'SingularTerm' option in `bvpset` to treat the singular term of the boundary-value problem appropriately.]

[6] (d) Verify that your leading-order approximation is indeed asymptotic by comparing the numerical and asymptotic solutions for $\epsilon \in [10^{-1}, 10^{-6}]$. Be sure to sufficiently resolve the boundary layer for all ϵ. Explain why your plot demonstrates asymptoticness.

4. **Multiple scales and WKB theory**

[8] (a) Why can WKB theory sometimes provide a good leading-order approximation even if there is no explicit small parameter in the ODE?

[12] (b) Consider the nonlinear oscillator

$$\ddot{y}(t) + \left(\omega^2(\epsilon t) + \epsilon y^2(t)\right) y(t) = 0,$$

with initial conditions $y(0) = 1$, $\dot{y}(0) = 0$. Use multiple-scale perturbation theory to find a leading-order approximation to $y(t)$ valid for $t = O(1/\epsilon)$.

[4] (c) Use `maple` to check your results for the drift in amplitude and phase when $\omega(s) = 1 + s\sin(s)$ and $\epsilon = 0.1$.

THE END