Assignment 4: Math744

1. $u'' + u = \varepsilon u^2 \quad , \quad u(0)=2, \quad u'(0)=0$

(a) try $u(t;\varepsilon) \sim u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t)$
(single time scale)

ε^0: $u_0'' + u_0 = 0 \quad ; \quad u_0(0)=2 \quad , u'(0)=0$

$\rightarrow \boxed{u_0(t) = 2\cos t}$

ε^1: $u_1'' + u_1 = u_0^2 = 4\cos^2 t \quad , \quad u_1(0) = u_1'(0) = 0$

$\rightarrow \boxed{u_1(t) = -\frac{4}{3}(\cos^2 t + \cos t - 2) = -\frac{4}{3}(\cos t - 1)(\cos t + 2)}$

(still okay)

ε^2: $u_2'' + u_2 = 2u_0 u_1 = -\frac{4^2}{3}\cos t\,(\cos^2 t + \cos t - 2)$

resonance
(secularity)

$\rightarrow u_2(t) = \frac{10}{3} t \sin t + \frac{2}{3}\cos t\left(\cos^2 t + \frac{8}{3}\cos t + \frac{5}{3}\right) - \frac{32}{9}$

$\therefore u(t) = 2\cos t - \varepsilon \frac{4}{3}(\cos^2 t + \cos t - 2)$

$+ \varepsilon^2\left[\frac{10}{3} t \sin t + \frac{2}{3}\cos t\left(\cos^2 t + \frac{8}{3}\cos t + \frac{5}{3}\right) - \frac{32}{9}\right] + O(\varepsilon^3)$

$\Rightarrow$ slow time scale $\boxed{T = \varepsilon^2 t}$

This solution is only valid for $t = O(1)$.

(b) consider two time scales: $t_0 = t$, $t_2 = \varepsilon^2 t$

$$\to \frac{d^2 u}{dt^2} = \frac{\partial^2 u}{\partial t_0^2} + \varepsilon^2 \, 2 \frac{\partial^2 u}{\partial t_0 \partial t_2} + \varepsilon^{4} \frac{\partial^2 u}{\partial t_2^2}$$

& let $u \sim u_0(t_0, t_2) + \varepsilon u_1(t_0, t_2) + \varepsilon^2 u_2(t_0, t_2)$

(note that we can formally add a time $t_1 = \varepsilon t$, but we would find zero variation on the time scale)

$$\varepsilon^0: \quad u_{0_{t_0 t_0}} + u_0 = 0, \quad u_0\big|_{t=0} = 2, \quad u_0'\big|_{t=0} = 0$$

$$\to u_0(t_0, t_2) = R(t_2) \cos(t_0 + \theta(t_2))$$

with $R(0) = 2$, $\theta(0) = 0$

$$\varepsilon^1: \quad u_{1_{t_0 t_0}} + u_1 = u_0^2 = (R\cos(t_0 + \theta))^2$$

$$\to \boxed{u_1(t_0, t_2) = \tfrac{1}{3} R^2 \left[\tfrac{1}{2} \sin 2t_0 \sin 2\theta + \cos^2 t_0 \sin^2\theta + \sin^2 t_0 \cos^2\theta + 1\right]}$$

$$+ \cancel{A_1(t_2)\sin t_0 + A_2(t_2)\cos t_0} \; R_1(t_2)\cos(t_0 + \theta_1(t_2))$$

$\cancel{\Rightarrow A_1 = A_2 = 0}$ $u_1\big|_{t=0} = 0 \Rightarrow \cancel{R_1 = -4/3 \quad \theta_1 = 0}$

$\cancel{u_1 = \tfrac{1}{3}R^2[\;] - \tfrac{4}{3}\cos t_0}$ $\quad \boxed{R_1 = \theta_1 = 0}$

$$\varepsilon^2: \quad u_{2_{t_0 t_0}} + u_2 = 2 u_0 u_1 - 2 u_{0_{t_0 t_2}}$$

$$= 2\left\{ \frac{R^3}{3} \cos(t_0 + \theta)[\quad] + R' \sin(t_0 + \theta) + R\theta' \cos(t_0 + \theta) \right\}$$

$-4/3 \cos t_0$

$$\to 2\left\{ R' \sin(t_0 + \theta) + \left(\frac{R^3}{3}[\quad] + R\theta' \right) \cos(t_0 + \theta) \right\}$$

$\Rightarrow R' = 0$ (eliminates $\sin(t_0+\theta)$ term) $\to \boxed{R = 2}$

after some simplification (using (convert, exp) & expand and simplify in maple) we find remaining RHS terms are

$$\underbrace{2/_3 \cos(3(t_0+\theta))}_{\text{not a problem.}} + 10/_3 \cos(t_0+\theta) + 2\theta' \cos(t_0+\theta)$$

∴ we would like $\frac{10}{3} + 2\theta' = 0$

$\Rightarrow \theta' = 5/3 \Rightarrow \boxed{\theta = -5/_3 T}$

∴ the zeroth order solution accurate to $t \sim \varepsilon^{-2}$ is:

(c)

$$\boxed{u(t;\varepsilon) = 2\cos\left(t\left(1 - 5/_3\,\varepsilon^2\right)\right)}$$

* no change in amplitude
* frequency shift of $-5/_3\,\varepsilon^2$

(4)

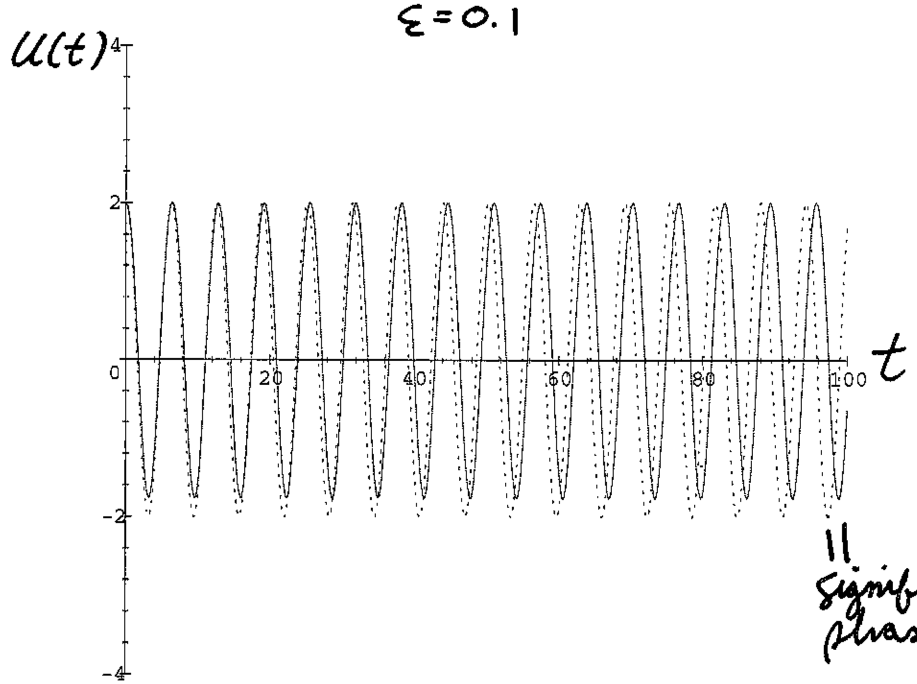
```
> with(plots):
> p:=dsolve({diff(u(t),t$2)+u(t)=0.1*u(t)^2,u(0)=2,D(u)(0)=0},u(t),type=numeric);
```

$p :=$ **proc**$(rkf45\_x)$ ... **end**

```
> plot1 := odeplot(p, [t,u(t)], 0..100, view=[0..100,-4..4], thickness=3, numpoints=1000):
> plot2 := plot(2*cos(t), t=0..100, view=[0..100,-4..4], linestyle=2, thickness=3,
  numpoints=1000):
> plot3 := plot(2*cos(t*(1-5/3*0.1^2)), t=0..100, view=[0..100,-4..4], linestyle=2,
  thickness=3, numpoints=1000):
> plots[display]({plot1,plot2});
```

$\varepsilon = 0.1$

$u(t) = 2\cos t$
secularity at
$t = O(\varepsilon^{2}) = O(100)$



‖
significant
phaseerror!

```
> plots[display]({plot1,plot3});
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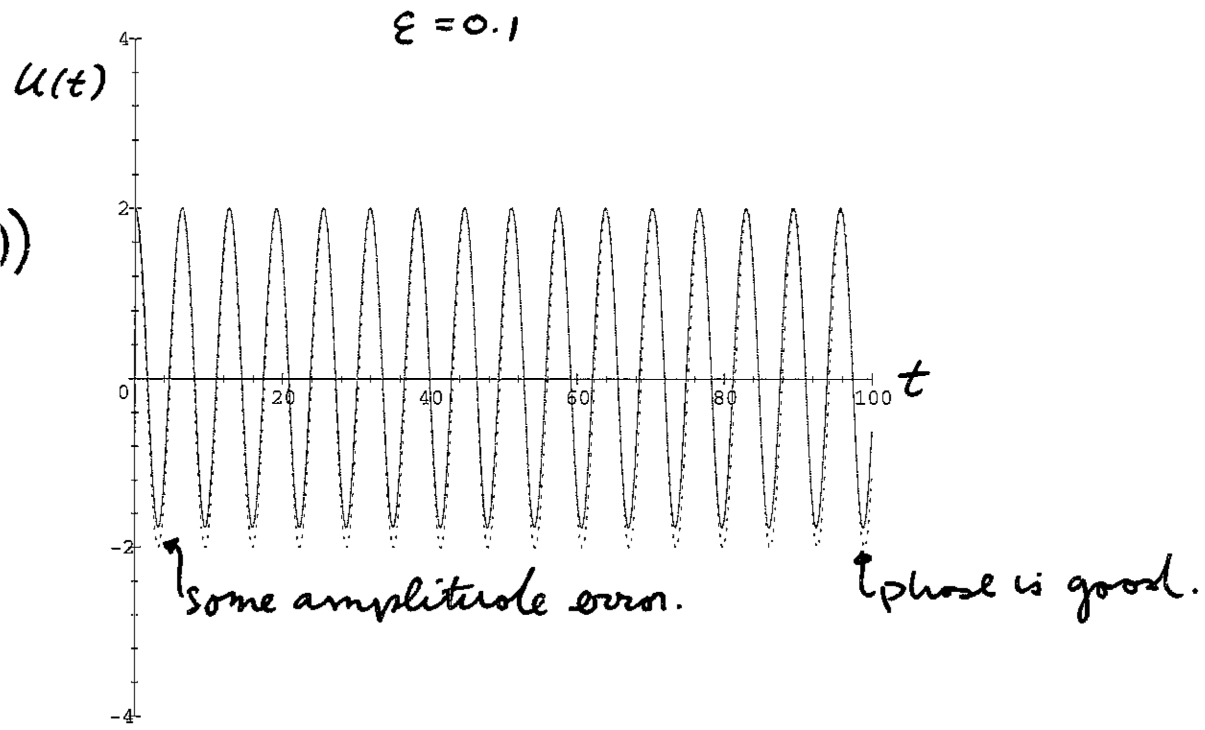
$u(t) = 2\cos\left(t\left(1 - \tfrac{5}{3}\varepsilon^2\right)\right)$
accurate to
$t = O(\varepsilon^{-2}) = 100$



[>

2. $\ddot{y} + \varepsilon y^2 \dot{y} + y = 0$, $y(0) = 1$, $\dot{y}(0) = 0$, $\varepsilon > 0$

try "naive" expansion first to determine when secularity appears. $y \sim y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$

ε^0: $\ddot{y}_0 + y_0 = 0 \rightarrow y_0(t) = \cos t$

ε^1: $\ddot{y}_1 + y_1 = -y_0^2 \dot{y}_0 = +\cos^2 t \sin t = \frac{1}{4}(\sin 3t + \sin t)$

secularity!

$\therefore$ use multiple scales analysis with

$t_0 = t$, $t_1 = \varepsilon t$.

$\rightarrow \frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1}$, $\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_0^2} + 2\varepsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2}{\partial t_1^2}$

& let $y(t;\varepsilon) \sim y_0(t_0, t_1) + \varepsilon y_1(t_0, t_1) + \varepsilon^2 y_2(t_0, t_1)$

$\underline{\varepsilon^0} \rightarrow y_{0_{t_0 t_0}} + y_0 = 0 \rightarrow \begin{cases} y_0(t_0, t_1) = R(t_1) \cos(t_0 + \theta(t_1)) \\ R(0) = 1, \ \theta(0) = 0. \end{cases}$

$\underline{\varepsilon^1}$: $y_{1_{t_0 t_0}} + y_1 = -y_0^2 y_{0_{t_0}} - 2 y_{0_{t_0 t_1}}$

$= R^3 \cos^2(t_0 + \theta) \sin(t_0 + \theta) + 2R' \sin(t_0 + \theta)$
$+ 2R\theta' \cos(t_0 + \theta)$

$= \frac{R^3}{4}\left(\sin 3(t_0+\theta) + \sin(t_0+\theta)\right) + 2R' \sin(t_0 + \theta) + 2R\theta' \cos(t_0 + \theta)$

not a problem

$\rightarrow$ want $\frac{R^3}{4} \sin(t_0 + \theta) + 2R' \sin(t_0 + \theta) + 2R\theta' \cos(t_0 + \theta)$

$= 0 \Rightarrow \theta' = 0 \rightarrow \boxed{\theta(t_1) = 0}$

and $R' = -\frac{R^3}{8} \rightarrow \boxed{R(t_1) = \frac{2}{\sqrt{t_1 + 4}}}$

$\therefore$ to leading order:

$$\boxed{y(t;\varepsilon) \sim \frac{2\cos t}{\sqrt{\varepsilon t + 4}}}$$

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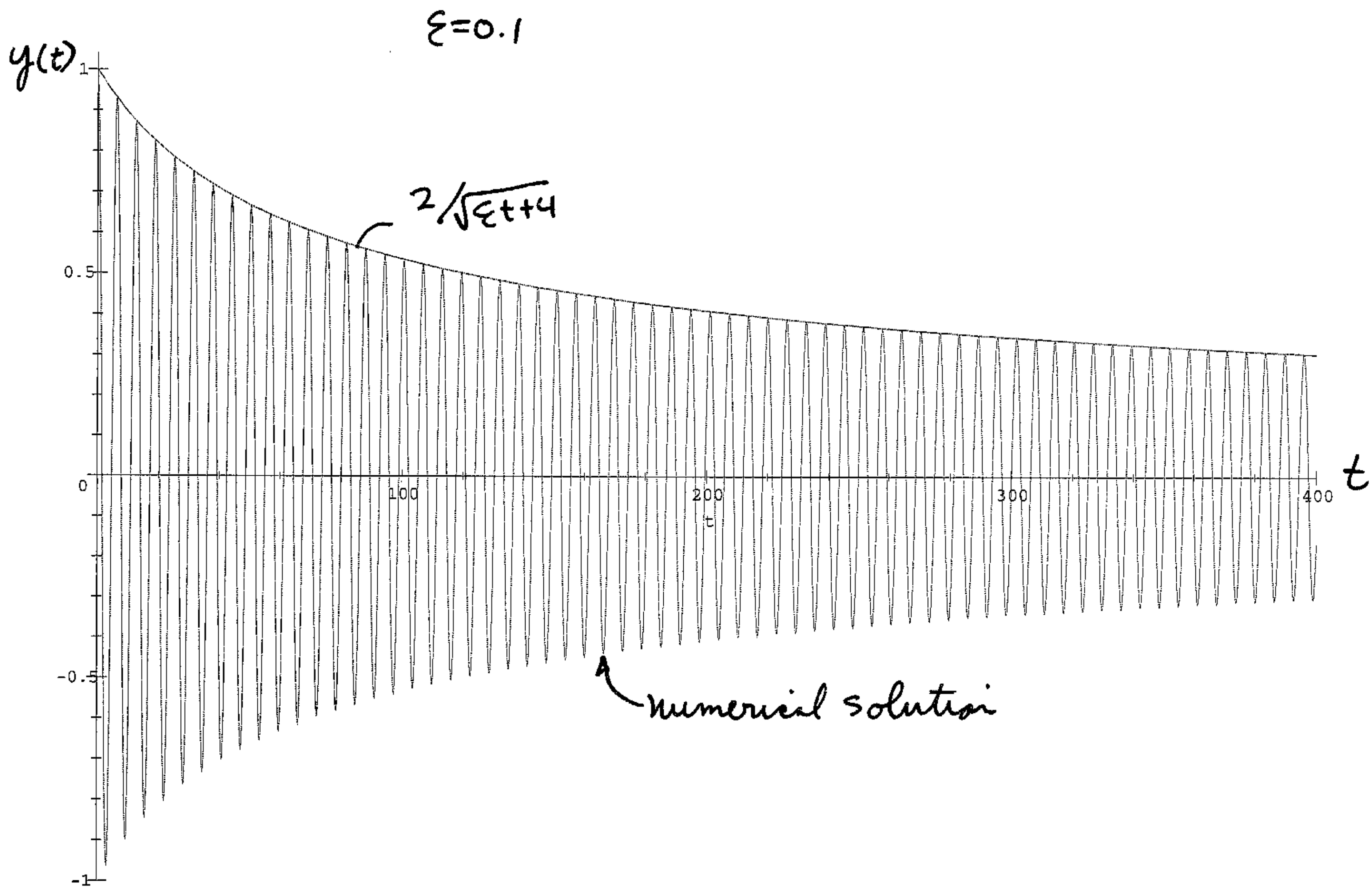
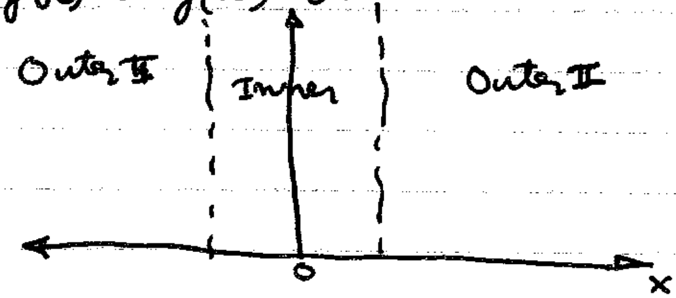


3.

$$\varepsilon^2 y''(x) = Q(x)\,y(x) \qquad \begin{cases} Q(x) = Q(-x) \quad \text{(even)} \\ Q(0) = 0, \ \lim_{x\to 0_+} Q'(x) = a \\ \lim_{x\to 0_-} Q'(x) = -a \end{cases}$$

$\cancel{Q(x) \sim a|x|}$

B.C. $\cancel{y(x)\to}$ $y(\infty) = 0$.



re-write eqn. : $\ddot{y} + f(\varepsilon t)\,y = 0 \qquad (f(\varepsilon t) = -Q(\varepsilon t))$

inner equation : $\boxed{\ddot{y} - \varepsilon a|t|\,y = 0}$

or $\ddot{y} - \varepsilon a t y = 0 \quad (t>0)$

$\ddot{y} + \varepsilon a t y = 0 \quad (t<0)$

or $\ddot{y} = \varepsilon a|t|y$ → exponential solutions only.

∴ inner solution has same form when matching to outer I & Outer II (both exp.).

outer I : $\dfrac{1}{[\varepsilon|t|a]^{1/4}}\Big[A\exp(\varphi) + B\exp(-\varphi)\Big]$

$\varphi = -\frac{2}{3}[\varepsilon a]^{1/2}\,|t|^{3/2}$

inner I: $\frac{1}{\tau^{1/4}\sqrt{\pi}}\left[\frac{1}{2}\alpha \exp\left(-\frac{2}{3}\tau^{3/2}\right) + \beta \exp\left(\frac{2}{3}\tau^{3/2}\right)\right]$

where $\tau = (\varepsilon a)^{1/3}|t|$

$\rightarrow \alpha = \frac{2\sqrt{\pi}}{[\varepsilon a]^{1/6}} A$, $\beta = \frac{\sqrt{\pi}}{[\varepsilon a]^{1/6}} B. = 0$ (since decaying)

outer II

$\frac{1}{[\varepsilon a|t|]^{1/4}}\left(a \exp(\theta) + b \exp(-\theta)\right)$

matching $\rightarrow$ $A = a$

& solution is $y(t) \sim A[+Q(\varepsilon t)]^{-1/4} \exp[-\theta]$

where. $\theta = 2/3 [\varepsilon a]^{1/2} |t|^{3/2}$

or $y(x) \sim A[Q(x)]^{-1/4} \exp[-\theta]$

$\theta = 2/3 [\varepsilon a]^{1/2} |x/\varepsilon|^{3/2}$ $x \to \pm\infty$

$y(t) \sim \frac{A}{[\varepsilon a]^{1/6}\tau^{1/4}} \exp\left(-2/3\, \tau^{3/2}\right)$

$t \to 0$.

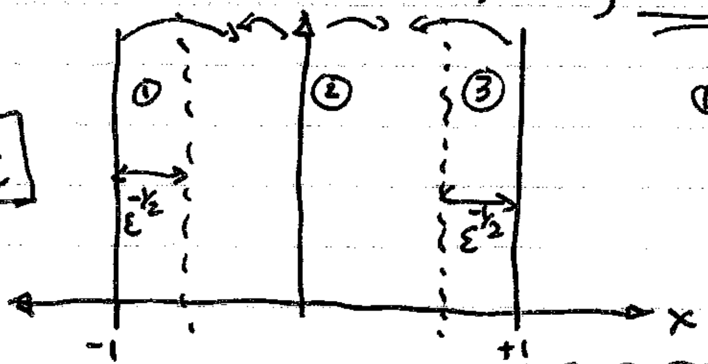
4. Find large e-value solutions of

$$y'' + \underbrace{\lambda(1-x^2)^2}_{\text{note freq.} > 0 \to \text{always oscillating.}} y = 0$$

B.C. $y(\pm 1) = 0$, assume $\lambda \gg 1$

Note at $x = \pm 1$ $Q(x) = 0$, $\underline{\underline{Q'(x) = 0}}$, $Q''(x) = 8$

$\boxed{\lambda^{-1} = \varepsilon}$



① + ③ are inner regions
② is outer region

Do analysis for case $Q(0) = 0$, $Q'(0) = 0$, $Q''(0) > 0$
for $\ddot{x} + Q(\varepsilon t)x = 0$
∴ near origin we have

$$\boxed{\ddot{x} + \frac{1}{2}(\varepsilon t)^2 Q''(0) x = 0}$$

let $\tau = t[\frac{1}{2}\varepsilon^2 Q''(0)]^{1/4}$ $(t = O(\varepsilon^{-1/2})$ in inner layer$)$

$$\to \boxed{x_{\tau\tau} + \tau^2 x = 0}$$

$\boxed{\text{near } x = -1 :}$ $Q(x) = (1-x^2)^2 \approx \frac{8}{2}(x+1)^2 = 4(x+1)^2$

Region ①

$\therefore y'' + \lambda 4(x+1)^2 y = 0$ → width $\lambda^{-1/4} \ll 1$

$\to$ let $\boxed{(x+1)(4\lambda)^{1/4} = \tau}$ → $(x+1)^2 = \dfrac{\tau^2}{(4\lambda)^{1/2}}$

$\dfrac{\partial^2}{\partial x^2} = \dfrac{\partial^2}{\partial \tau^2}\dfrac{\partial \tau^2}{\partial x^2} = (4\lambda)^{1/2}$

limit $\tau \to +\infty$

$$\to (4\lambda)^{1/2} y_{\tau\tau} + \frac{4\lambda\,\tau^2}{(4\lambda)^{1/2}} y = 0 \to \boxed{y_{\tau\tau} + \tau^2 y = 0}$$

B.C. $\boxed{y(0) = 0}$

Region 3
similarly, near $x=+1$:

$$\boxed{y_{\sigma\sigma} + \sigma^2 y = 0 \quad , -(x-1)(4\lambda)^{1/4} = \sigma}$$

B.C. $\boxed{y(0)=0}$

overlap : $\sigma \longrightarrow +\infty$

Region 2 : WKB approx. with $Q(x) = \lambda(1-x^2)^2$

express ans in τ & σ & take $\lim_{\sigma\to\infty}$, $\lim_{\tau\to\infty}$.

& match with inner solutions.

$$y(x) \sim [\lambda(1-x^2)^2]^{-1/4}(a\sin\theta + b\cos\theta)$$

$$\theta = \int_x^1 \lambda^{1/2}(1-t^2)\,dt \quad \text{or} \quad \int_{-1}^{x} \lambda^{1/2}(1-t^2)\,dt$$

$\therefore$ as $x\to -1$ $\quad \theta_- \sim \lambda^{1/2}(1+x)^2$; $x\to +1$ $\quad \theta_+ \sim \lambda^{1/2}(1-x)^2$

~~y(x) ~~~

$\therefore$ as $x\to -1$ $\quad y(x) \sim \lambda^{-1/4}\, 2^{-1/2}(1+x)^{-1/2}[a\sin\theta_- + b\cos\theta_-]$

$x\to +1$ $\quad y(x) \sim \lambda^{-1/4}\, 2^{-1/2}(1-x)^{-1/2}[a\sin\theta_+ + b\cos\theta_+]$

Solution of $y_{\tau\tau} + \tau^2 y = 0$; $y(0) = 0$.

$$y(\tau) = c_1 \tau^{1/2} J_{1/4}\left(\tfrac{1}{2}\tau^2\right)$$

$$\therefore \lim_{\tau\to\infty} y(\tau) = c_1 \frac{2}{\sqrt{\pi}} \tau^{-1/2} \sin\left(\frac{1}{2}\tau^2 + \frac{\pi}{8}\right)$$

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① outer : $y(x) \sim \lambda^{-1/4} 2^{-1/2} (1+x)^{-1/2} [a \sin(\theta_-) + b\cos(\theta_-)]$

inner $y(x) \sim \frac{2^{3/4}}{\sqrt{\pi}} \lambda^{-1/8} (1+x)^{-1/2} \sin\Big(\underbrace{\lambda^{1/2}(1+x)^2}_{\theta_-} + \frac{\pi}{8}\Big)$

~~② : outer : $y(x) \sim \lambda^{-1/4} 2^{-1/2} (1-x)^{-1/2} [a \sin\theta_+ + b\cos\theta_+]$~~

~~inner $y(x) \sim \frac{2^{3/4}}{\sqrt{\pi}} \lambda^{-1/8} (1-x)^{-1/2} \sin\Big(\underbrace{\lambda^{1/2}(1-x)^2}_{\theta_+} + \frac{\pi}{8}\Big)$~~

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① $\rightarrow$ $\frac{2^{3/4}}{\sqrt{\pi}} \lambda^{-1/8} \sin\left(\theta_- + \frac{\pi}{8}\right) = \lambda^{-1/4} 2^{-1/2} [a \sin\theta_- + b\cos\theta_-]$

$$\rightarrow \frac{2^{5/4} \lambda^{1/8}}{\sqrt{\pi}} \sin\left(\theta_- + \frac{\pi}{8}\right) = a\sin\theta_- + b\cos\theta_-$$
$$= A\sin(\theta_- + \delta)$$

~~② $\rightarrow$ $\frac{2^{5/4} \lambda^{1/8}}{\sqrt{\pi}} \sin\left(\theta_+ + \frac{\pi}{8}\right) = a\sin\theta_+ + b\cos\theta_+$~~
~~$= A\sin(\theta_- + \delta)$~~

$$\rightarrow \boxed{A = \frac{2^{5/4}\lambda^{1/8}}{\sqrt{\pi}}, \quad \delta = \frac{\pi}{8}}$$

$$\therefore \boxed{y(x) \sim \lambda^{-1/8}(1-x^2)^{-1/2}\frac{2^{5/4}}{\sqrt{\pi}} \sin\left(\int_{-1}^{x}\lambda^{1/2}(1-t^2)dt + \frac{\pi}{8}\right)}$$

Now, determine λ using B.C. at $x=+1$, ie require

$\sin\left(\int_{x}^{1}\lambda^{1/2}(1-t^2)\,dt + \frac{\pi}{8}\right)$ to have same form

as $\sin\left(\int_{-1}^{x}\lambda^{2}(1-t^2)\,dt + \frac{\pi}{8}\right)$

$$\rightarrow \sin\left\{\int_{-1}^{x}\lambda^{1/2}(1-t^2)\,dt + \frac{\pi}{8}\right\} = \sin\left\{-\left[\int_{x}^{1}\lambda^{1/2}(1-t^2)\,dt + \frac{\pi}{8}\right]\right.$$
$$\left. + \left[\int_{-1}^{1}\lambda^{1/2}(1-t^2)\,dt + \frac{\pi}{4}\right]\right\}$$

$$\rightarrow \text{require} \quad \int_{-1}^{1}\lambda^{1/2}(1-t^2)\,dt + \frac{\pi}{4} = n\pi$$

$(\sin(A+B) = \sin A\cos B + \cancel{\sin}\cos A\sin B)$

$$\rightarrow \quad \tfrac{4}{3}\lambda^{1/2} + \frac{\pi}{4} = n\pi$$

$$\rightarrow \tfrac{4}{3}\lambda^{1/2} = n\pi - \pi/4 = \pi(n - 1/4)$$

$$\rightarrow 16/9\,\lambda = \pi^2(n^2 - \tfrac{1}{2}n + 1/16)$$

$$\rightarrow \lambda = \frac{9\pi^2}{16}(n^2 - \tfrac{1}{2}n) + 1/16\cdot\pi^2\frac{9}{16}$$

$$\rightarrow \boxed{\lambda = \frac{9\pi^2}{16}n(n-\tfrac{1}{2}) + \frac{\pi^2 9}{(16)^2}}, \quad n = 0, 1, 2, \ldots$$

$$\text{or } \boxed{\lambda = \frac{9\pi^2}{16}(n - 1/4)^2}$$

5. $$\varepsilon^2 y''(x) = [1+(\sin x)^2]\, y(x)$$

WKB Approx. $y(x) \sim \exp\left[\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n(x)\right]$

Physical Optics : $y(x) \sim \exp\left[\frac{1}{\varepsilon} S_0(x) + S_1(x)\right]$, $\varepsilon \to 0$

$\therefore$ For Physical Optics to be a good approx. we require

$$\varepsilon |S_2(x)| \ll 1 \quad , \quad \varepsilon \to 0.$$

Now, $S_2(x) = \int^x \frac{Q''}{8Q^{3/2}} - \frac{5}{32}\frac{Q'^2}{Q^{5/2}}\, dt$

integrand $= \frac{1}{4}\left[\frac{\cos 2x}{(1+\sin^2 x)^{3/2}} - \frac{5}{8}\frac{\sin 2x}{(1+\sin^2 x)^{5/2}}\right]$

but, $\frac{1}{1+\sin^2 x} \leq 1$

$\to$ integrand $\leq \frac{1}{4}\left[\cos 2x - \frac{5}{8}\sin 2x\right]$

$\to \varepsilon S_2(x) \leq \frac{1}{4}\int^x \cos 2t - \frac{5}{8}\sin 2t\, dt$

$\leq \varepsilon \frac{3}{4^3} \sin 2x$

$\to \varepsilon |S_2(x)| \leq \varepsilon \frac{3}{4^3} |\sin 2x| \leq \varepsilon \frac{3}{4^3} \quad \forall x$

& $\therefore \varepsilon |S_2(x)| \leq \varepsilon \frac{3}{4^3} \ll 1$ as $\varepsilon \to 0 \ \forall x$

$\therefore$ WKB is a good approx. to $y(x)$ as $\varepsilon \to 0 \ \forall x$ and is accurate as $x \to \infty$

Now, check asymptoticness: we require

$$\varepsilon|S_2(x)| \ll |S_1(x)| \ll \frac{1}{\varepsilon}|S_0(x)|$$

$$|S_0(x)| = \left|\int^x \underbrace{\sqrt{1+\sin^2 x'}}_{\geq 1}\,dx'\right| \geq x \; \cancel{\geq \int^x \sin x'\,dx' = -\cos x}$$

$$|S_1(x)| = \left|\tfrac{1}{4}\ln\left(1+\tfrac{1}{2}\sin^2 x\right)\right| \leq \tfrac{1}{4}\ln 2$$

∴ taking upper bound for S_1 + lower bound for S_0 (worst case) we have

$$\frac{1}{4}\ln 2 \ll \frac{1}{\varepsilon}|x| \quad \text{okay! (note that if } x=0 \text{ then } S_1 = 0\text{ !)}$$

similarly upper bound for S_1 (from prev. page) and ~~lower bound for~~ use actual $S_1(x)$

$$\varepsilon|S_2(x)| \qquad\qquad |S_1(x)|$$

$$\varepsilon\,\frac{3}{4^3}|\sin 2x| \ll \frac{1}{4}\left|\ln(1+\sin^2 x)\right| \quad \forall x \; \cancel{\text{except } x=0}$$

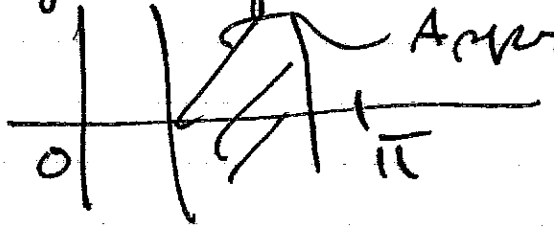
~~only problem at $x = n\pi$, $n = 0, \pm 1, \pm 2$,~~

Problems at $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

~~since approx. is okay for $\forall x$ (except $x=0$) there is no point taking more terms. taking more terms will not improve $x=0$.~~

~~Approx valid for one period~~ until $x=\pi$ ~~, then re-start~~.

Must stay away from $x = n\pi$



Approx okay here.

Taking more terms will not help.