Assignment 4 **MATH 744** **18 November 2010**

This assignment is due 2 December in class.

1. Consider the following perturbation problem:

$$\frac{d^2u}{dt^2} + u = \epsilon u^2 \text{ as } \epsilon \to 0$$

with initial conditions $u(0) = 2, u'(0) = 0$.

[4] (a) In what order of ϵ does a secularity first appear in the regular perturbation solution for $u(t)$? Find $u(t)$ to that order in ϵ in this expansion.

[6] (b) Introduce a suitable slow time scale to eliminate the secularity you found in (a). [Hint: use `maple` commands to convert the trigonometric functions to exponential form, then expand, and finally simplify to help find the form of resonant terms.]

[4] (c) Find the zeroth order solution on the slow time scale. What are the amplitude and frequency shifts? Use `maple` to plot the numerical solution compared with the regular perturbation expansion and the leading-order multiple scales solution.

2. Consider the following nonlinear initial value problem,

$$\ddot{y} + \epsilon y^2 \dot{y} + y = 0; \qquad y(0) = 1, \qquad \dot{y}(0) = 0, \qquad \epsilon > 0.$$

[4] (a) Find the leading-order multiple-scales approximation to the initial value problem. What is the slow time scale?

[4] (b) Use `maple` to compare the amplitude variation of the leading-order solution to the numerical solution.

[10] 3. Solve $\epsilon^2 y''(x) = Q(x)y(x)$, where $Q(x)$ is even, $Q(x)$ vanishes just once at $x = 0$, and $Q(x) \sim a|x|$ near $x = 0$ ($a > 0$). Find that solution y which vanishes as $x \to \infty$.

[10] 4. Find the large eigenvalue solutions of the equation

$$y'' + \lambda(1 - x^2)^2 y = 0$$

subject to $y = 0$ at $x = \pm 1$.

At the ends $x = \pm 1$ you will need to use turning point solutions like

$$(1 - x^2)^{1/2} J_{1/4}(\lambda^{1/2}(1 - x^2)/4),$$

and then use $J_{1/4}(z) \sim (2/\pi z)^{1/2} \cos(z - 3\pi/8)$ as $z \to \infty$.

[8] 5. Consider the following equations:

$$\epsilon^2 y''(x) = [1 + (\sin x)^2] y(x).$$

For which fixed values of x is the usual two-term WKB approximation a good approximation to $y(x)$ as $\epsilon \to 0$? Is this approximation accurate as $x \to \infty$? Can the approximation be improved by taking more terms?

Total: 50