Assignment 3 - Math 744

1. (a) $x^3 - x^2 + \varepsilon = 0$

what is the dominant balance as $\varepsilon \to 0$?

(i) suppose $x^3 \sim \varepsilon \Rightarrow x = O(\varepsilon^{1/3})$

this is inconsistent since then $x^2 = O(\varepsilon^{2/3}) > \varepsilon$ and would dominate

(ii) suppose $x^2 \sim \varepsilon \Rightarrow x = O(\varepsilon^{1/2})$

okay since $x^3 = O(\varepsilon^{3/2}) < \varepsilon$.

This suggests that the roots have the series form

$$x = \sum_{n=0}^{\infty} \varepsilon^{n/2} a_n$$

thus this is a singular perturbation problem in the sense that as $\varepsilon \to 0$ the roots tend to their unpert. values much more slowly, i.e. like $O(\varepsilon^{1/2})$ (the $x=0$ root splits into $\pm$ pair, the $x=1$ root behaves like a regular perturbation problem.)

also, for any finite ε $x=0$ gives 2 distinct roots whereas $\varepsilon=0$ gives a degenerate root

(b) Suppose $x = \sum_{n=0}^{\infty} a_n \varepsilon^{n/2}$

ε^0: $a_0^3 - a_0^2 = 0 \rightarrow x_1 = 0, \; x_2 = 1$

$x_1 = 0 \rightarrow$ let $x_1 = 0 + \sum_{n=1}^{\infty} a_n \varepsilon^{n/2}$ + set equal powers to 0

ε^1: $a_1^2 = 1 \rightarrow \boxed{a_1 = \pm 1}$ root splits at this order.

$\rightarrow \boxed{a_{1a} = +1, \; a_{1b} = -1}$

$\varepsilon^{3/2}$: $a_1^3 = 2a_1 a_2 \rightarrow \begin{cases} a_{2a} = 1/2 \\ a_{2b} = 1/2 \end{cases}$

ε^2: $\cancel{5/4 = 2a_3}$ $\rightarrow \begin{cases} a_{3a} = 5/8 \\ a_{3b} = -5/8 \end{cases}$

$\varepsilon^{5/2}$: $\begin{cases} a_{4a} = 1 \\ a_{4b} = 1 \end{cases}$

so to $O(\varepsilon^2)$

$$\boxed{\begin{aligned} x_{1a} &= 0 + \varepsilon^{1/2} + 1/2\,\varepsilon + 5/8\,\varepsilon^{3/2} + \varepsilon^2 \\ x_{1b} &= 0 - \varepsilon^{1/2} + 1/2\,\varepsilon - 5/8\,\varepsilon^{3/2} + \varepsilon^2 \end{aligned}}$$

and x_2 : ("regular")

let $x_2 = 1 + \sum_{n=0}^{\infty} \varepsilon^{n/2} a_n$

$\underline{\varepsilon^{1/2}}$: $a_1 = 0$, ε^1: $a_2 = -1$, $\underline{\varepsilon}^{3/2}$: $a_3 = 0$, ε^2: $a_4 = -2$

$$\boxed{x_2 = 1 - \varepsilon - 2\varepsilon^2}$$

---

2. (a) $\begin{cases} \varepsilon y'' + y y' - y = 0, & 0 \le x \le 1 \\ y(0) = 0, \quad y(1) = 3 \end{cases}$

[note that this is a nonlinear problem, but it should work out!]

Assume B.C. at $x=0$:

$\therefore$ $\dfrac{x}{\varepsilon^\alpha} = \xi$ what is the scaling? $0 < \alpha$

$\rightarrow$ $\dfrac{\varepsilon}{\varepsilon^{2\alpha}} y'' + \dfrac{1}{\varepsilon^\alpha} y y' - y = 0$

$\rightarrow$ $y'' + \varepsilon^{\alpha-1} y y' - \varepsilon^{2\alpha-1} y = 0$

Now, "outer" ~~solution~~ eqn. is $yy' - y = 0$ and we must ~~overlap~~ have at least one of these terms in inner equation at leading order to have overlaps:

$\alpha = 1 \rightarrow$ $y'' + yy' - \cancel{\varepsilon y} = 0$

contains y'' & overlaps $\rightarrow$ $\boxed{\alpha = 1}$ okay inner solution

let $\quad x/\varepsilon = \xi \quad \rightarrow \quad y'' + yy' - \varepsilon y = 0$

& $\quad y = \sum_{n=0}^{\infty} \varepsilon^n y_n$

leading order: $\quad y_0'' + y_0 y_0' = 0 \quad , \; y_0(0) = 0$

$$\rightarrow \frac{y_0''}{y_0'} = -y_0$$

using Maple we find that a solution satisfying the B.C. at $x=0$ is

$$y_0(\xi) = 2c_1 \tanh(c_1 \xi)$$

call this $\boxed{y_I(\xi) = 2c_1 \tanh(c_1 \xi)}$

outer solution $\quad$ let $y_0(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$

leading order is: $\quad y_0 y_0' - y_0 = 0 \quad , \quad y_0(1) = 3$

$$\rightarrow \boxed{y_0(x) = x + 2}$$

in overlap, $\boxed{x \rightarrow 0, \; \xi \rightarrow \infty}$

$$\rightarrow y_0 = 2 = 2c_1 \quad \rightarrow \quad \boxed{c_1 = 1}$$

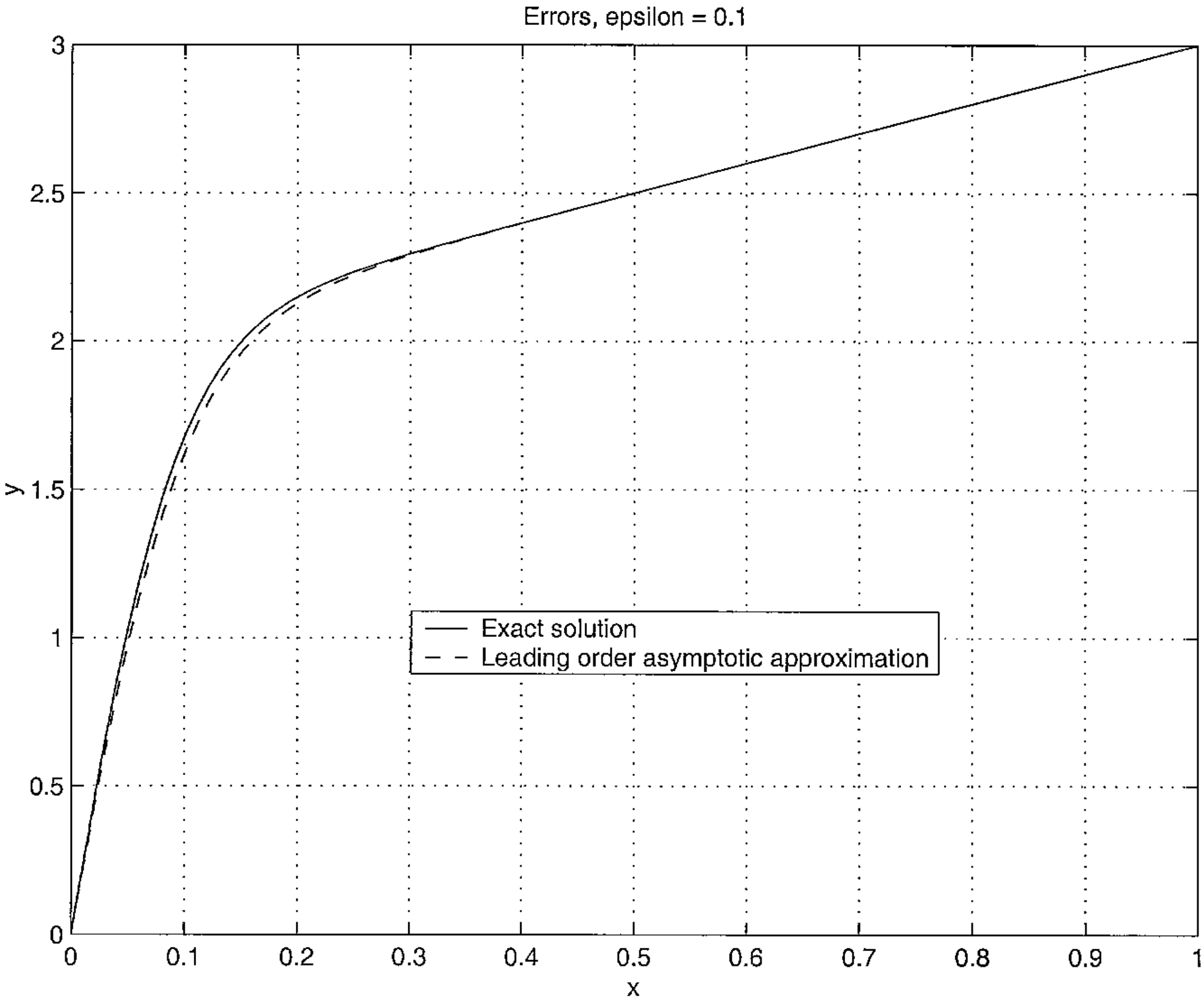
$\therefore$ outer solution is $\; y_I(x) = 2\tanh(x/\varepsilon)$

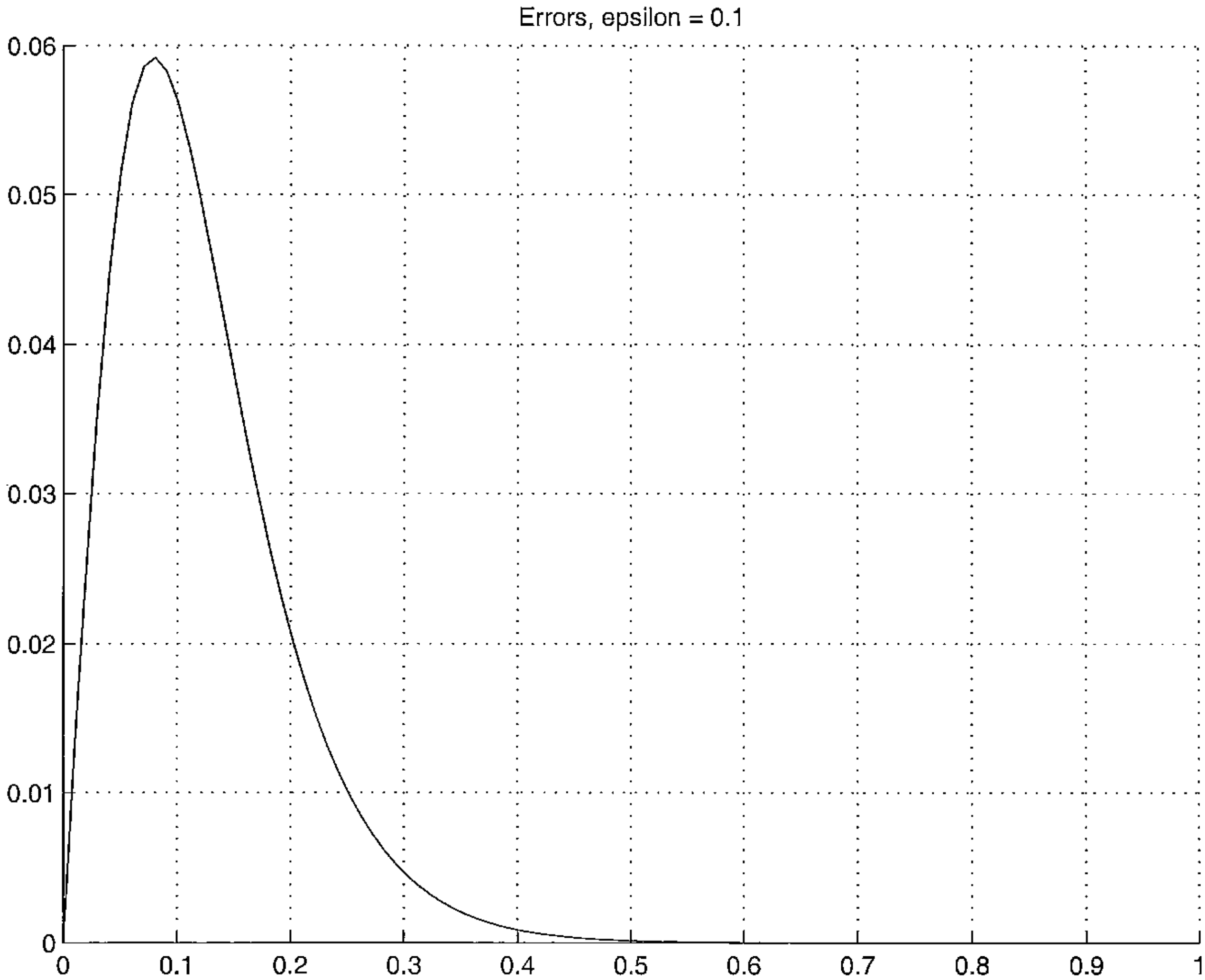
& uniform approx. is:

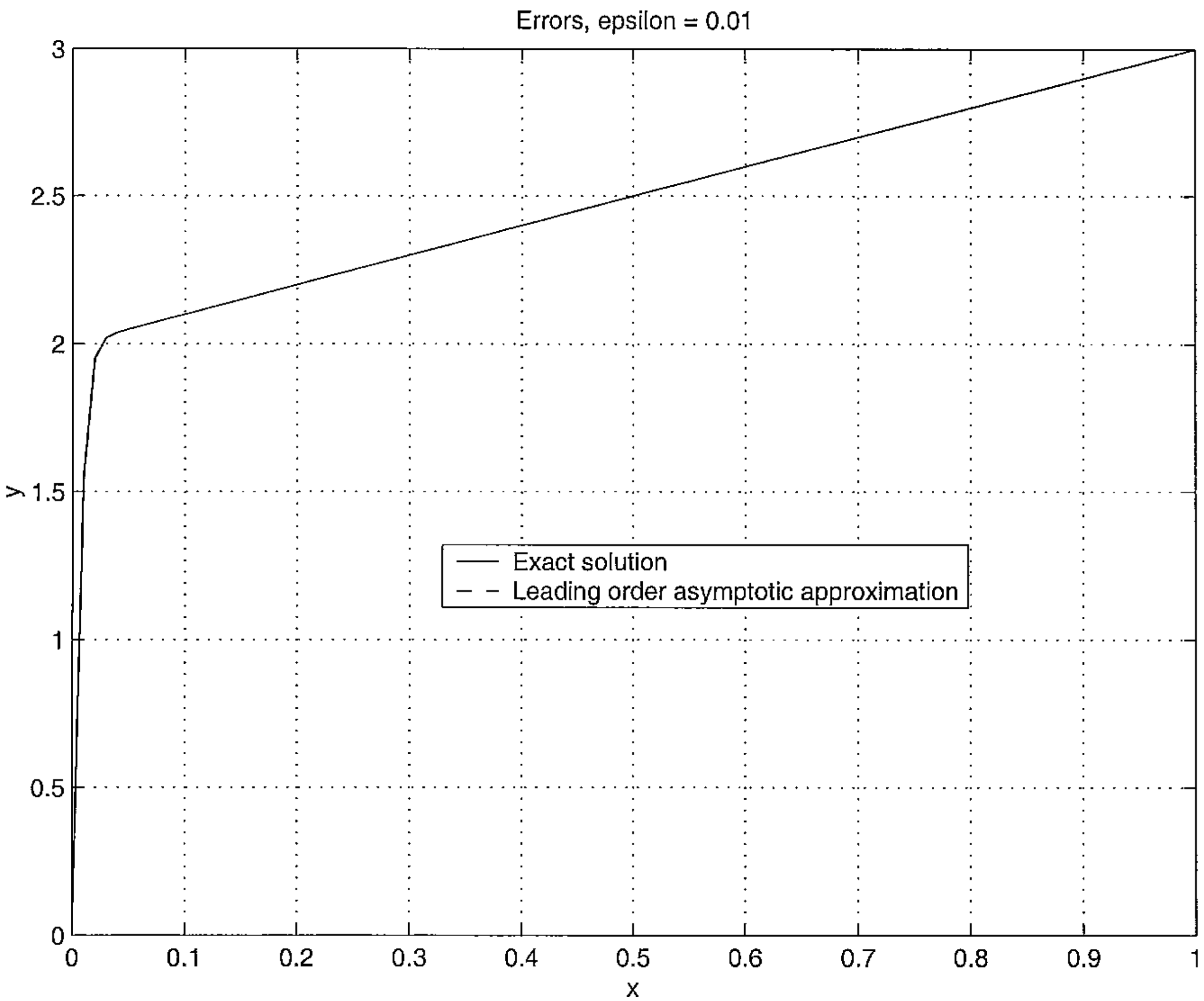
$$y_{unif}(x) = (x+2) + 2\tanh(x/\varepsilon) - 2$$

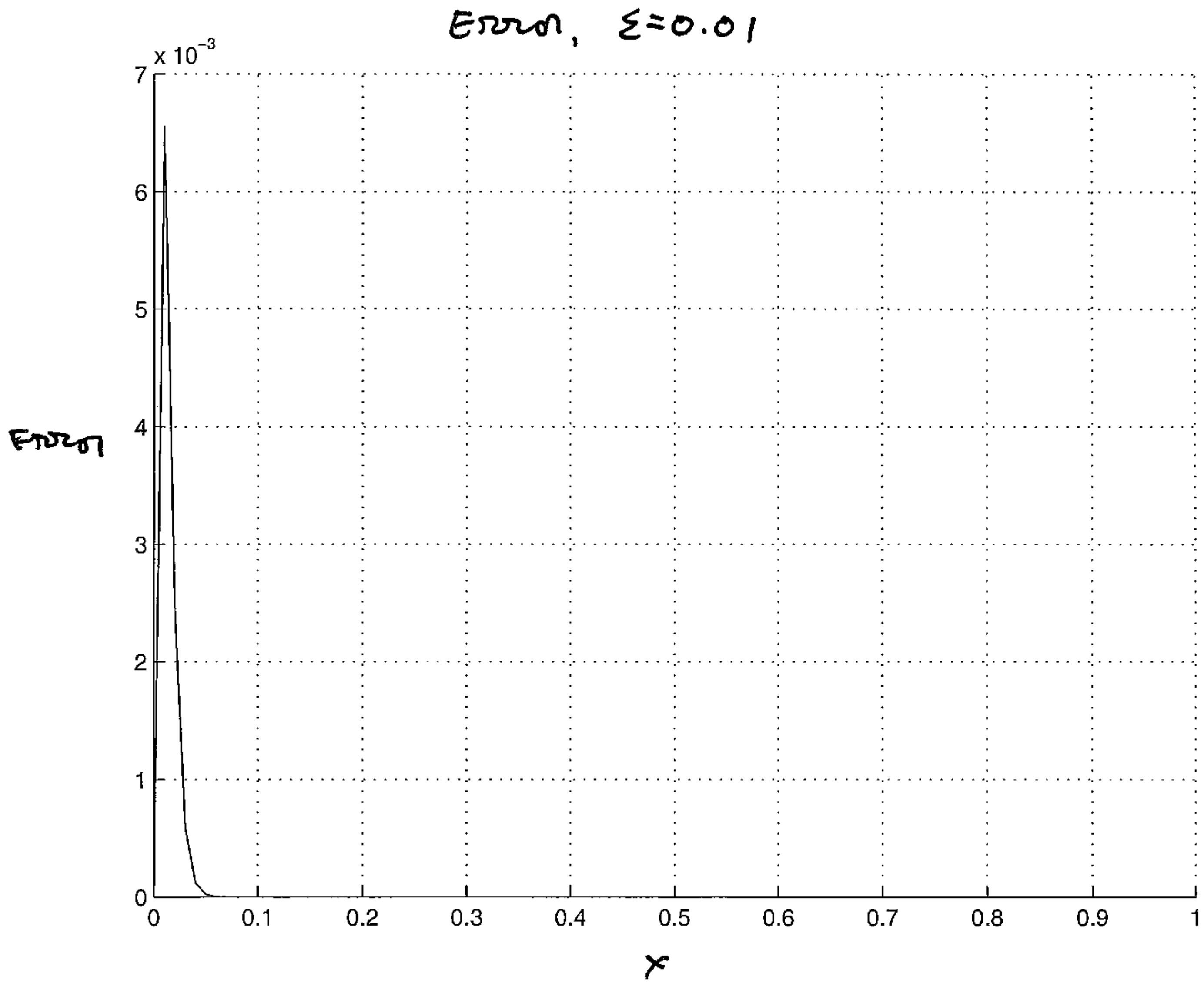
or $\boxed{y_{unif}(x) = x + 2\tanh(x/\varepsilon)}$

(b) (see attached figure)


Errors, epsilon = 0.1
3
2.5
2
1.5
1
0.5
0
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
Exact solution
Leading order asymptotic approximation




Errors, epsilon = 0.01
3
2.5
2
y 1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
Exact solution
Leading order asymptotic approximation


Error, ε=0.01
x 10^-3
7
6
5
4
3
2
1
0
Error
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

3. $\varepsilon y'' + x^{\alpha} y' + y = 0, \quad y(0) = y(1) = 1$ as $\varepsilon \to 0_+$

To have a boundary layer at $x=0$ need a <u>balance</u> between inner & outer solutions.

<u>Outer solution</u> : (eqn) $x^{\alpha} y' + y = 0$, B.C. $y(1) = 1$

<u>Inner solution</u> : assume scaling $x/\varepsilon^{\beta} = \xi$, $0<\beta$
sub in eqn:

$$\frac{\varepsilon}{\varepsilon^{2\beta}} y'' + \frac{\varepsilon^{\alpha\beta}\xi^{\alpha}}{\varepsilon^{\beta}} y' + y = 0$$

or $$y'' + \varepsilon^{\beta(\alpha+1)-1} \xi^{\alpha} y' + \varepsilon^{2\beta-1} y = 0$$

need to retain either y' or y term at leading order in ε & need retained term to be same order ~~at~~ as y'', i.e. $O(1)$.

$\rightarrow \beta = 1/2$ (1) or $\beta(\alpha+1) = 1$ (2)

(1): $\beta = 1/2 \rightarrow$ y' term is $O\left(\varepsilon^{\frac{1}{2}(\alpha+1)-1}\right)$
want $\frac{1}{2}(\alpha+1) - 1 > 0 \rightarrow \frac{1}{2}(\alpha+1) > 1 \rightarrow (\alpha+1) > 2$
$\rightarrow \boxed{\alpha > 1}$

---

(2) $\beta(\alpha+1) = 1 \rightarrow \beta = 1/(\alpha+1) \rightarrow$ y-term is $O\left(\varepsilon^{2/(\alpha+1)-1}\right)$

want $2/(\alpha+1) - 1 > 0 \rightarrow 2/(\alpha+1) > 1$
$\rightarrow (\alpha+1) < 2 \rightarrow \boxed{\alpha < 1}$

---

(3) $\boxed{\alpha=1}$ $y'' + \varepsilon^{2(\beta-\frac{1}{2})}\xi y' + \varepsilon^{2\beta-1} y = 0$ | (a) $\beta = 1/2$ all terms equal, $2\beta \gtrless 1 \rightarrow y''=0 \rightarrow$ No B.L.

$\therefore$ we can have B.L. for $0<\alpha<1$ & $\alpha>1$
thickness is $O(\varepsilon^{1/2})$ for $\alpha \geq 1$ & $O(\varepsilon^{1/(\alpha+1)})$ for $0<\alpha<1$ ... For $\alpha<0$ see next page...

$\boxed{\alpha<0}$ $\varepsilon y'' + \frac{1}{x^\alpha} y' + y = 0$, $y(0) = y(1) = 1$, $\alpha > 0$

let $\frac{x}{\varepsilon^\beta} = \xi \Rightarrow y'' + \frac{\varepsilon^{\beta(1-\alpha)}}{\xi^\alpha} y' + \varepsilon^{2\beta-1} y = 0$

y dominant: $\rightarrow \beta = 1/2$ & require $\frac{1}{2}(1-\alpha) - 1 > 0$
$\rightarrow 1-\alpha > 2 \rightarrow \alpha < -1$ X can't be satisfied since $\alpha > 0$!

y' dominant: $\beta(1-\alpha) - 1 = 0 \rightarrow \boxed{\beta = \frac{1}{1-\alpha}}$

but need $\beta > 0$ for B.L. $\rightarrow \boxed{\alpha < 1}$
check that $2\beta - 1 > 0 \rightarrow \frac{2}{1-\alpha} > 1 \rightarrow$ true if $\alpha < 1$ ✓

If $\boxed{\alpha = 1}$ $y'' + \frac{\varepsilon^{-(1+\alpha)}}{\xi^\alpha} y' + \varepsilon^{2\beta-1} y = 0$

no good since y' dominates y''.

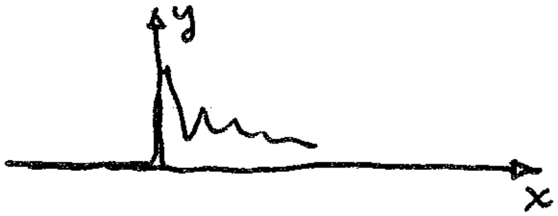
In summary:
$O(\varepsilon^{1/2})$ for $\alpha \geq 1$! (*) see below!
B.L. $O(\varepsilon^{1/1+\alpha})$ for $0 \leq \alpha < 1$
B.L. $O(\varepsilon^{\frac{1}{1-|\alpha|}})$ for $-1 < \alpha < 0$

(*) Question: do we really have a B.L. for $\alpha \geq 1$?
No: since inner solⁿ is oscillatory with $y_i(x) \propto \cos\left(\frac{x}{\varepsilon^{1/2}}\right)$ we actually have a multiplescales type problem where the frequency is $O(\varepsilon^{-1/2})$ as $x \to 0$, but damping dominates as $x \to 1$. Use WKB by changing variables $t = x/\varepsilon^{1/2}$
$\Rightarrow y'' + \varepsilon^{\frac{1}{2}(\alpha-1)} t^\alpha y' + y = 0$ & let $\varepsilon^{\frac{1}{2}(\alpha-1)} = \delta^{\alpha+1} \Rightarrow \ddot{y} + \delta(\delta t)^\alpha \dot{y} + y = 0$
which is general WKB form $\ddot{y} + \varepsilon a(\delta t)\dot{y} + b(\delta t) y = 0$ with $a(\delta t) = (\delta t)^\alpha$, $b(\delta t) = 1$. N.B. $\alpha = 1$ is WKB with $\varepsilon = 1$!

4. $\varepsilon y'' + a(x) y' + b(x) y = \delta(x)$, $y(x) = 0$ $x < 0$

to <u>leading order</u> in ε as $\varepsilon \to 0+$

$a(x) > 0$ for $x \geq 0$



$a(x) > 0$ for $x \geq 0 \Rightarrow$ B.L. near $x = 0$.

let $\xi = x/\varepsilon^\alpha$, $\alpha > 0$

$$\to \frac{\varepsilon}{\varepsilon^{2\alpha}} y'' + \frac{a(\varepsilon^\alpha \xi)}{\varepsilon^\alpha} y' + b(\varepsilon^\alpha \xi) y = \delta(\varepsilon^\alpha \xi)$$

$$\Rightarrow y'' + \varepsilon^{\alpha-1} a(\varepsilon^\alpha \xi) y' + \varepsilon^{2\alpha-1} b(\varepsilon^\alpha \xi) y = \varepsilon^{2\alpha-1} \delta(\varepsilon^\alpha \xi)$$

leading order $y'' + \varepsilon^{\alpha-1} a(0) y' + \varepsilon^{2\alpha-1} b(0) y = \varepsilon^{2\alpha-1} \delta(0)$

<u>Outer Solution</u>:

$$y' + \frac{b(x)}{a(x)} y = \frac{\delta(x)}{a(x)} \qquad (a(x) > 0)$$

$$\to y(x) = c_1 \exp\left[-\int_0^x \frac{b(t)}{a(t)} dt\right] + \exp\left[-\int_0^x \frac{b(t)}{a(t)} dt\right] \int_0^x \frac{\delta(t)}{a(t)} \exp\left[\int_0^t \frac{b(w)}{a(w)} dw\right]$$

$$\to y(x) = \exp\left[-\int_0^x \frac{b(t)}{a(t)} dt\right] \left\{ c_1 + \frac{1}{a(0)} \right\} H(x)$$

↳ Heaviside

(see p. 14 ~~Kron~~ BO).

c_1 determined by matching to inner solution $(x \to 0)$

Inner Solution

$\alpha = 1$ gives balance ($\alpha = 1/2 \Rightarrow y'$ term dominates)

$\therefore$ to leading order

$$y'' + a(0)\, y' = \underbrace{\varepsilon f(\varepsilon \xi)}_{\text{keep since } f \text{ is large at } \xi = 0}$$

soln: $y_I(\xi) = \dfrac{H(\xi)}{a(0)} \left[1 - e^{-a(0)\xi} \right] + d_1 + d_2 e^{-a_0 \xi}$

But, $y(\xi) = 0$ for $\xi < 0 \Rightarrow d_1 = d_2 = 0$

and $\boxed{y_I(\xi) = \dfrac{H(\xi)}{a(0)} \left[1 - e^{-a(0)\xi} \right]}$

Matching: outer $(x \to 0)$: $C_1 + \dfrac{1}{a(0)}$

inner $(\xi \to \infty)$: $\dfrac{1}{a(0)}$

$\rightarrow C_1 = 0$

$$\therefore \boxed{y(x) = \frac{H(x)}{a(0)} \left[\exp\left[-\int_0^x \frac{b(t)}{a(t)}\, dt \right] - e^{-a(0)\xi} \right]}$$

5. (a) $\varepsilon y'' + 2x y' - 4x^2 y = 0, \quad -1 \le x \le 1 \quad y(-1) = 0$
$y(+1) = e$

Note : $a(x) = 2x = 0$ at $x = 0$ (an internal pt.) !

$\therefore$ expect an internal B.L. near $x=0$.
& outer solutions away from $x=0$.

Inner: suppose $\xi = x/\varepsilon^\alpha$, $\alpha > 0$, $x = \varepsilon^\alpha \xi$

$$\to \frac{\varepsilon}{\varepsilon^{2\alpha}} y'' + \frac{2\cancel{\varepsilon^\alpha}\xi}{\cancel{\varepsilon^\alpha}} y' - 4\varepsilon^{2\alpha}\xi^2 y = 0$$

$$\to y'' + \varepsilon^{2\alpha-1} 2\xi y' - \varepsilon^{4\alpha-1} 4\xi^2 y = 0$$

y' dominant $\to \alpha = 1/2 \to$ y term $O(\varepsilon)$ ✓
y " $\quad \alpha = 1/4 \to$ y' term $O(\varepsilon^{-1/2})$ × (y' dominates)

$\therefore \alpha = 1/2 \to \xi = x/\varepsilon^{1/2}$ & leading order is

$$y'' + 2\xi y' = 0 \to \boxed{y_I(\xi) = C_1 + C_2 \operatorname{erf}(\xi)}$$

(Maple) $\cancel{y_I(\xi) = C_1 \xi^{1/2} J_{1/3}\left(\frac{2\sqrt{2}}{3}\xi^{3/2}\right) + C_2 \xi^{1/2} Y_{1/3}\left(\frac{2\sqrt{2}}{3}\xi^{3/2}\right)}$

Outer I : near $x = +1$ B.C. $y(+1) = e$

$$\to 2x y' - 4x^2 y = 0 \to y' - 2xy = 0$$

$$\to \boxed{y_{OI}(x) = \exp(x^2)}$$

Outer II : near $x = -1$ B.C. $y(-1) = 0$.

$$\to y' - 2xy = 0 \to \boxed{y_{OII}(x) = 0}$$

Match: $x \to 0, \quad \xi \to \pm\infty$

Outer I match: $x \to 0_+, \quad \xi \to +\infty$

Outer: $y_{OI}(x) \sim 1$

Inner: $y_{I\infty}(\xi) \sim c_1 + c_2 \quad \to \boxed{c_1 + c_2 = 1}$

Outer II match: $x \to 0_-, \quad \xi \to -\infty$

Outer: $y_{OII}(x) = 0$ (as always!)

Inner: $y_I(\xi) \sim c_1 - c_2 \quad \to \boxed{c_1 = c_2}$

Combining info from both boundaries: $c_1 = c_2 = \frac{1}{2}$

$\therefore$ solution is: $y(x) = y_{OI} + \cancel{y_{OII}} + y_I - \cancel{2}\, y_{over1} - \cancel{y_{over2}}$

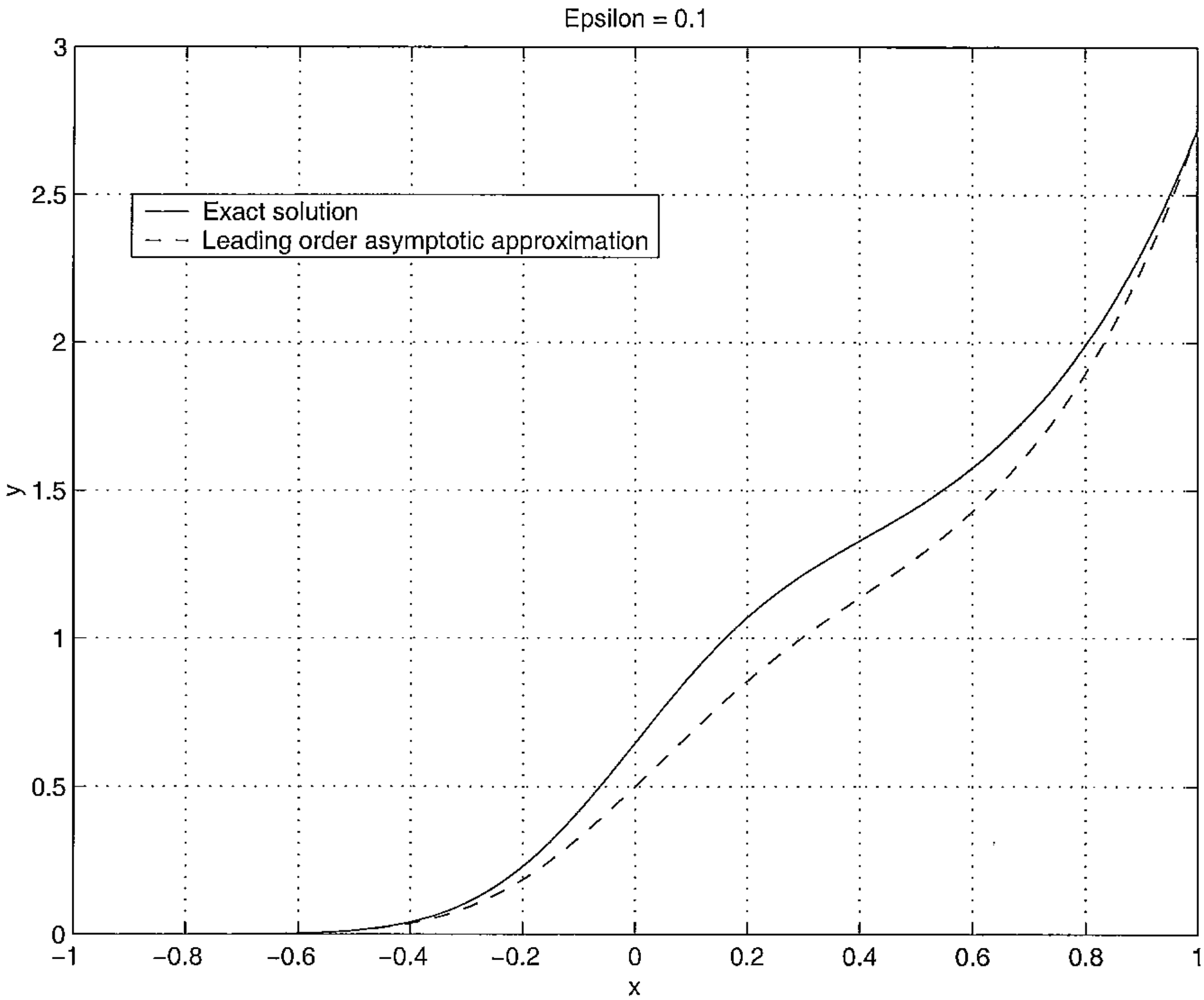
$x>0$

$\to y(x) = \exp(x^2) + 0 + \frac{1}{2}[1 + \mathrm{erf}(x/\varepsilon^{1/2})] - 1$

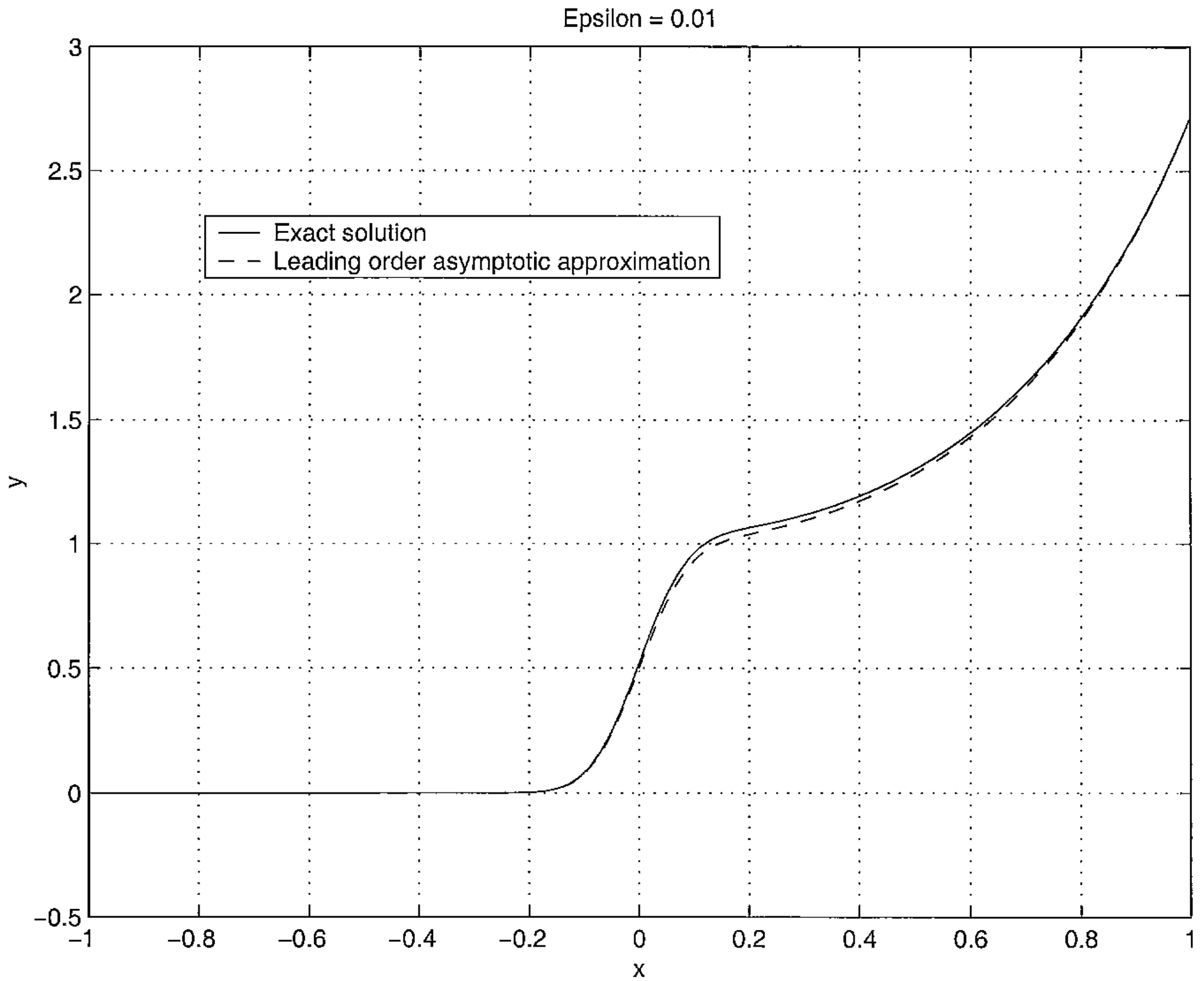
$$\to \boxed{y(x) = \exp(x^2) + \frac{1}{2}[\mathrm{erf}(x/\varepsilon^{1/2}) - 1]}\ , \quad x>0$$

$$\boxed{y(x) = \tfrac{1}{2}[\mathrm{erf}(x/\varepsilon^{1/2}) + 1]}\ , \quad x<0$$

or $$\boxed{y(x) = (e^{x^2} - 1)H(x) + \frac{1}{2}\left(\mathrm{erf}(x/\varepsilon^{1/2}) + 1\right)}$$



Epsilon = 0.1
Exact solution
Leading order asymptotic approximation
y
x
3
2.5
2
1.5
1
0.5
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



Epsilon = 0.01
Exact solution
Leading order asymptotic approximation
y
x
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1