Assignment 2 - Math 744

1. $$I(x) = \int_0^1 \sin\left[x\left(t + \frac{1}{6}t^3 - \sinh t\right)\right]\cos t\, dt$$

$$= \operatorname{Im}\int_0^1 \exp\Big[ix\underbrace{\left(t+\frac{1}{6}t^3 - \sinh t\right)}_{\psi(t)}\Big]\cos t\, dt$$

$x \to \infty$

$\psi'(t) = 1 + \frac{1}{2}t^2 - \cosh t$

$\psi'(0) = 1 + 0 - 1 = 0 \to$ stationary pt. $t_c = 0$

$\psi''(t) = t - \sinh t$

$\psi''(0) = 0 - 0 = 0$

$\psi'''(t) = 1 - \cosh t$

$\psi'''(0) = 0$

$\psi^{(4)}(t) = -\sinh(t)$

$\psi^{(4)}(0) = 0$

$\psi^{(5)}(t) = -\cosh(t)$

$\psi^{(5)}(0) = -1 \quad \to \quad \boxed{p=5}$

$\cos(0) = 1 \to f(a) = 1, \quad \psi(0) = 0$

$$\therefore\ I(x) \sim \operatorname{Im}\left[\exp\left[-i\,\pi/10\right]\left[\frac{5!}{x}\right]^{1/5}\frac{\Gamma(1/5)}{5}\right]$$

or $$I(x) \sim -\sin\left(\frac{\pi}{10}\right)\frac{(120)^{1/5}}{5}x^{-1/5}\,\Gamma(1/5)$$

$$\boxed{\cancel{I(x) \sim 24\,\Gamma(1/5)\sin(\pi/10)\,x^{-1/5}}}$$

$$\boxed{I(x) \sim -\frac{(120)^{1/5}}{5}\sin(\pi/10)\,\Gamma(1/5)\,x^{-1/5}}$$

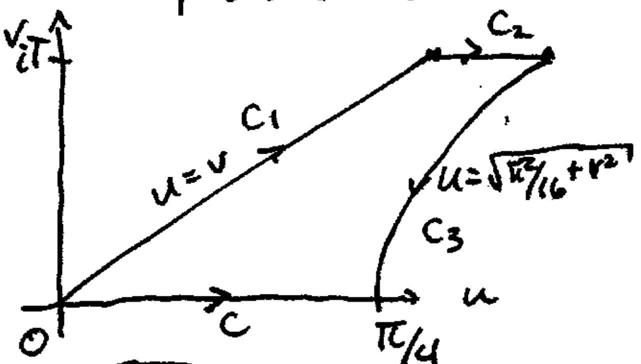
2. $I = \int_0^{\pi/4} \cos(xt^2)\tan^2 t\,dt \quad , x \to +\infty$

$= \mathrm{Re}\left[\cancel{\tfrac{1}{2}}\int_{\cancel{-\pi/4}\ 0}^{\pi/4} e^{ixt^2}\tan^2 t\,dt\right]$

$\to \rho(t) = it^2 = i(u+iv)^2 = i[(u^2-v^2)+i2uv]$
$= -2uv + i(u^2-v^2)$

constant phase contours have $u=v$ or $u=-v$

take $u=v$ for steepest descent



at $t=\pi/4 = \pi/4 + i\cdot 0$

$\mathrm{Im}(\rho(t)) = \frac{\pi^2}{16}$

$\to u^2 - v^2 = \pi^2/16$ (constant phase)

$\to u = \sqrt{\pi^2/16 + v^2}$

C_3 : $t = \sqrt{v^2+\pi^2/16} + iv$, $0 \le v < \infty$

C_2 : connects $(\pi/4 + iT)$ to $\sqrt{v^2+\pi^2/16} + iv$

contribution from C_2 vanishes: (as curves are asymptotic).

$$\int_{C_1} = (1+i)\int_0^\infty e^{-x2v^2}\underbrace{\tan^2((1+i)v)}_{\text{expand for } v \text{ small}}\,dv$$

$(1+i)\tan^2((1+i)v) \sim -2(1-i)v^2 + O(v^4)$

$\cancel{(1+i)v \mp \tfrac{2}{3}(1-i)v^3}$

$= \cancel{2i\int_0^\infty v e^{-x2v^2}dv \mp \tfrac{4}{3}\int_0^\infty v^3 e^{-x2v^2}dv}$

~~want real part~~ $\cancel{= -\frac{1}{24}\frac{1}{x^2}}$

$\to$ Real part is $\sim -2\int_0^\infty v^2 e^{-x2v^2}dv = -\frac{\sqrt{2\pi}}{8}\frac{1}{x^{3/2}}$ ✓

$$\int_{C_3} : \quad \rho(t) = -2v\sqrt{v^2+\pi^2/16} + i\pi^2/16$$

use <u>Watson's lemma</u>.

$\to$ let $\rho(t) = i\pi^2/16 - s = it^2 = i\pi^2/16 - 2v\sqrt{v^2+\pi^2/16}$

$\to s = 2v\sqrt{v^2+\pi^2/16} > 0$

$$\frac{dt}{} = \frac{\frac{1}{2}i\,ds}{\sqrt{\pi^2/16+is}}$$

$$\to \int_{C_3} = \frac{1}{2}ie^{ix\pi^2/16}\int_0^\infty \frac{\tan^2\left(\sqrt{\pi^2/16+is}\right)e^{-xs}}{\sqrt{\pi^2/16+is}}\,ds$$

change in direction!

$$\ominus\frac{4}{2\pi}e^{ix\pi^2/16}\int_0^\infty\left[i+\cancel{\frac{8}{\pi^2}\left(\frac{1}{\pi}-1\right)s}\right]e^{-xs}\,ds$$

$$\to -\frac{2}{\pi}\left(i\cos\pi^2 x/16 - \sin\pi^2 x/16\right)\left[\frac{1}{x}+\cancel{\frac{8i}{\pi}\left(\frac{1}{\pi}-1\right)\frac{1}{x^2}}\right]$$

Real part: $+\frac{2}{\pi}\sin x\pi^2/16\,\frac{1}{x} + \cancel{\frac{16}{\pi^2}\left(\frac{1}{\pi}-1\right)\frac{1}{x^2}\cos^2\pi^2/16}$

$\therefore$ Total:

$$\boxed{I(x) \sim +\frac{2}{\pi}\sin x\pi^2/16\,\frac{1}{x} - \frac{\sqrt{2\pi}}{8}\frac{1}{x^{3/2}}} + \cancel{\left[+\frac{16}{\pi^2}\left(\frac{1}{\pi}-1\right)x\cos\pi^2/16\right]\frac{1}{x^2}}$$

3. $$I(x) = \int_0^{1/e} \frac{e^{-xt}}{\ln t}\,dt, \quad x \to +\infty$$

let $r = xt \Rightarrow dr = x\,dt$, $t = r/x$ $\left(\begin{array}{l} r = O(1) \\ \to t = O(1/x) \\ \text{main contribution} \\ \text{to integral} \end{array}\right)$

$$I = \frac{1}{x}\int_0^{x/e \to \infty} \frac{e^{-r}\,dt}{\ln r - \ln x}$$

Now, expand in inverse powers of $\frac{1}{\ln x}$:

$$\frac{1}{\ln r - \ln x} = \frac{-1}{\ln x}\,\frac{1}{1 - \underbrace{\ln r/\ln x}_{\text{small}}} = \frac{-1}{\ln x}\sum_{n=0}^{\infty}\left(\frac{\ln r}{\ln x}\right)^n$$

$$\Rightarrow I(x) \sim \frac{-1}{x\ln x}\sum_{n=0}^{\infty}\int_0^{\infty}\left(\frac{\ln r}{\ln x}\right)^n e^{-r}\,dr$$

$$\sim -\frac{1}{x\ln x}\sum_{n=0}^{\infty}\frac{1}{(\ln x)^n}\int_0^{\infty}(\ln r)^n e^{-r}\,dr$$

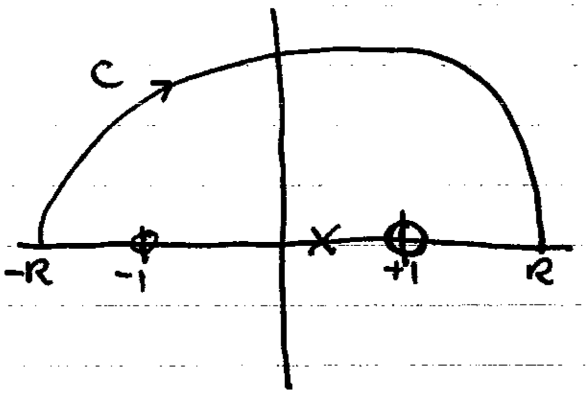
$\therefore$ leading-order term is

$$\sim \frac{-1}{x\ln x} \quad ✓$$

Note unusual expansion in powers of $\frac{1}{\ln x}$.

4. $$\int_C \frac{e^{ix(t^3/3-t)}}{t-a}\,dt, \quad x \to +\infty$$

simple pole at $t = +a$ (on real axis)



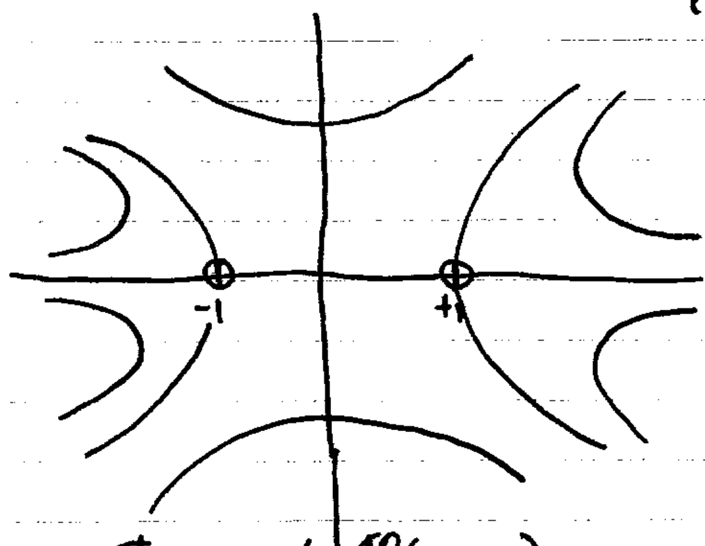
consider limit $R \to \infty$

$\times$ = pole
O = saddle pt.

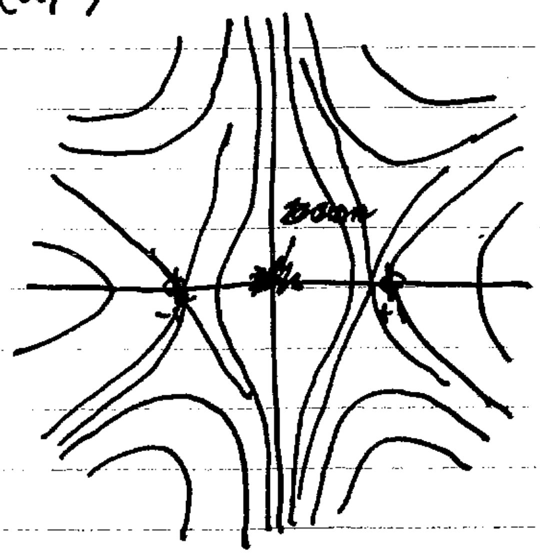
$$\rho(t) = i\left(t^3/3 - t\right), \quad \rho'(t) = i(t^2-1) \Rightarrow t_c = \pm 1$$
$$\rho''(t) = i2t \qquad \rho''(t_c) = \pm 2i$$

what are steepest phase contours?

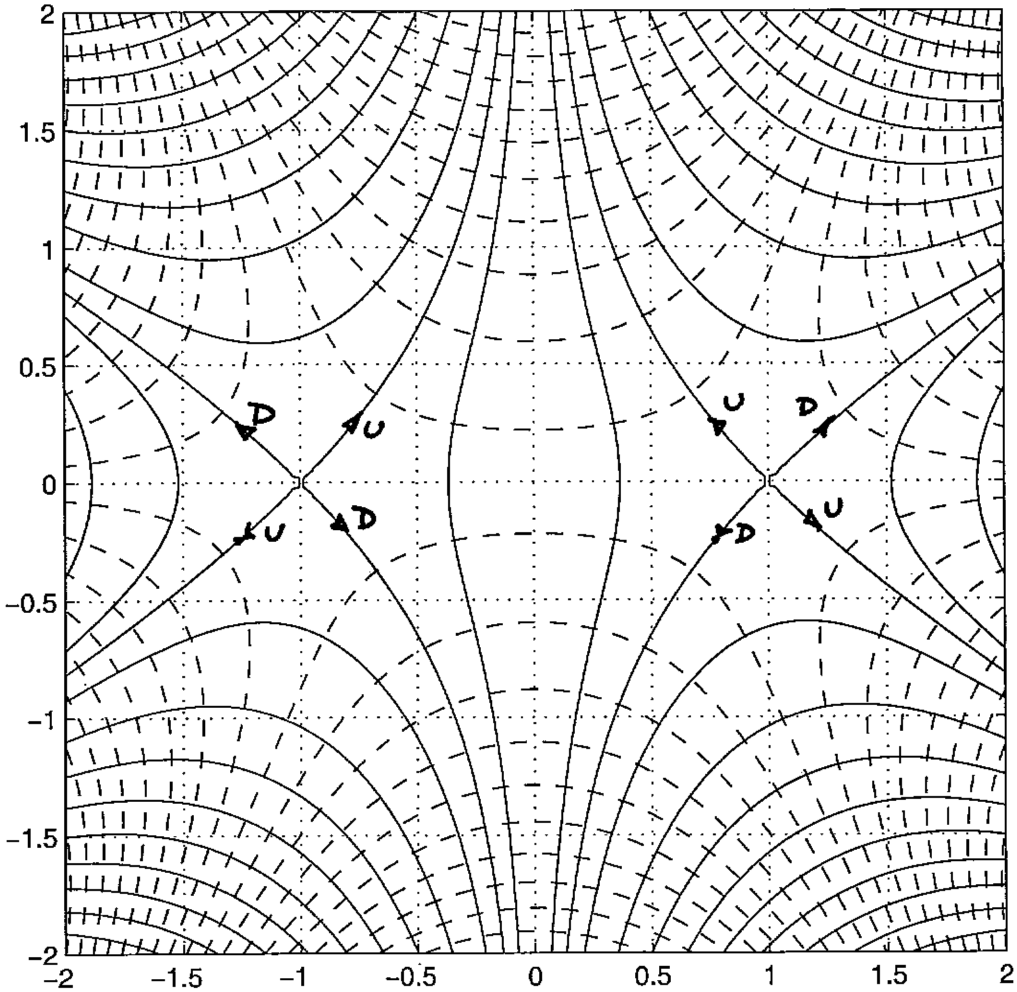
$$\rho(t) = i\left(\tfrac{1}{3}(u+iv)^3 - u - iv\right)$$
$$= \underbrace{iu\left[\tfrac{1}{3}u^2 - v^2 - 1\right]}_{\psi(u,v)} - \underbrace{v\left[u^2 - \tfrac{1}{3}v^2 - 1\right]}_{\varphi(u,v)}$$



contours of $\varphi(u,v)$



contours of $\psi(u,v)$
see print out

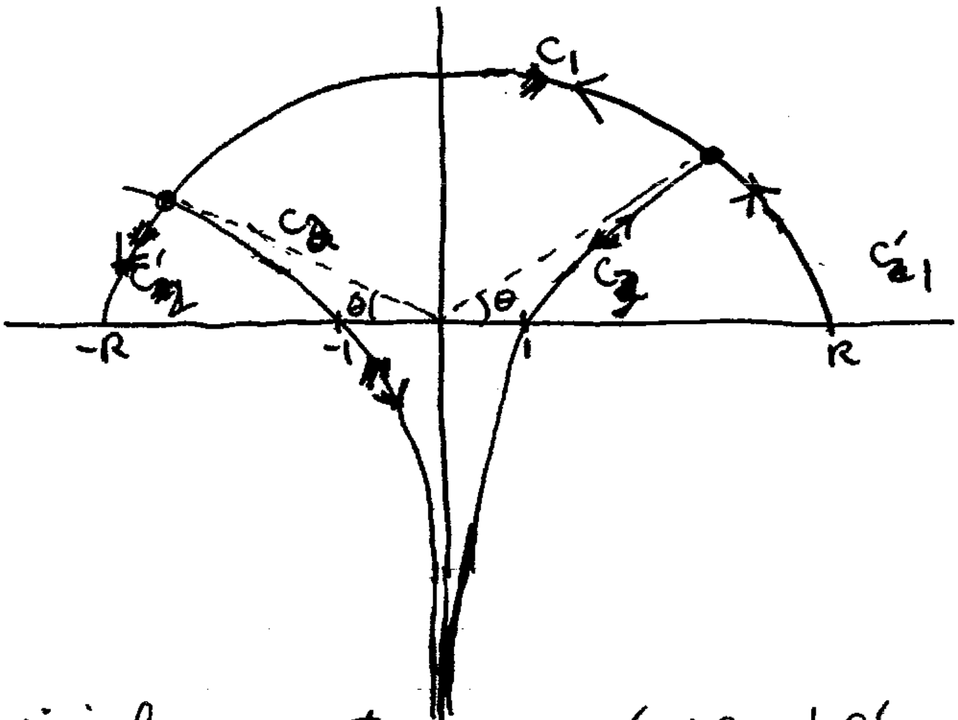


- - - - contours of φ

——— " " ψ

4. (cont.) (ii)

Consider the following deformation: (new contour) ~~that passes through~~ saddle points.



The original ~~con~~ contour $C = C_1' + C_1 + C_2'$, $\lim_{R\to\infty}$
and θ is the angle of asymptote of the steepest descent contours:

$$v = \sqrt{\frac{\frac{1}{3}u^3 - u + 2/3}{u}} \to v = \frac{1}{\sqrt{3}}u \text{ as } R\to\infty \ (u\to\infty) \to \boxed{\theta = \pi/6}$$

But contribution from $C_2' + C_1'$ vanish in limit $R\to\infty$:

$$\int_{C_2'} \le \frac{\pi/6}{2\pi}\cdot 2\pi R \cdot \frac{\exp(-\frac{1}{3}x R^3 \sin 3\theta)}{R} \quad \text{as } R\to\infty$$

but $0 \le \sin 3\theta \le 1$ if $0 \le \theta \le \pi/6$, ie $\sin 3\theta \ge 0$

∴ $\lim_{R\to\infty} \int_{C_2'} = 0$ and sim. for $\int_{C_1'}$.

so $-\int_C = \int_{C_1}$ ∴

$$-\int_C + \int_{C_2} + \int_{C_3} = \begin{cases} 0 & \text{if } |a|>1 \\ 2\pi i\, e^{ix(a^3/3 - a)} & \text{if } |a|<1 \\ \pi i\, e^{ix(a^3/3 - a)} & \text{if } a = \pm 1 \end{cases}$$

(use Cauchy principal value)

4. (iii)

The integrals $\int_{C_2} + \int_{C_3}$ are dominated by the contribution from the saddle pt. at $t = \pm 1$

$\int_{C_3}$: contour near $t=1$ may be approximated by the straight line

$$t \sim (1+i)s + 1 \qquad (s \in \mathbb{R})$$
$$dt \sim (1+i)\,ds$$

[sub in $t = 1 + re^{i\theta}$ in $\rho(t)$ expand for small r & set $\rho(r^2)$ imaginary part $= 0$]

$$\Rightarrow e^{ix(t^3/3 - t)} \sim e^{-2/3\,ix}\, e^{-2s^2 x}$$
$$\Rightarrow \frac{1}{t-a} \sim \frac{1}{1-a}$$

$$\therefore \int_{C_3} \sim \frac{(1+i)}{(1-a)} e^{-i\,2/3\,x} \int_{-\infty}^{\infty} e^{-2s^2x}\,ds$$

$$\sim \frac{(1+i)}{(1-a)} \sqrt{\frac{\pi}{2}}\, e^{-i\,2/3\,x}\, x^{-1/2}$$

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similarly for $\int_{C_2}$:

contour may be approximated by $t \sim (1-i)s - 1$

$$\Rightarrow dt \sim +(1-i)\,ds$$
$$e^{ix(t^3/3 - t)} \sim e^{+i\,2/3\,x}\, e^{-2s^2x}$$
$$\frac{1}{t-a} \sim \frac{-1}{(1+a)}$$

$$\therefore \int_{C_2} = \frac{-(1-i)}{(1+a)} e^{i\,2/3\,x} \sqrt{\frac{\pi}{2}}\, x^{-1/2}$$

$$\therefore \int_{C_2} + \int_{C_3} = \sqrt{\frac{\pi}{2}}\, x^{-1/2} \left[\frac{(1+i)}{(1-a)} e^{-i\,2/3\,x} - \frac{(1-i)}{(1+a)} e^{+i\,2/3\,x} \right]$$

4. (iv)

<u>To resume</u>:

(i) if $|a|>1$ singularity is <u>outside</u> curve &

$$\int_c = \sqrt{\frac{\pi}{2}}\, x^{-1/2}\left[\frac{(1+i)}{(1-a)}\, e^{-i\,2/3\,x} - \frac{(1-i)}{(1+a)}\, e^{+i\,2/3\,x}\right]$$

(ii) if $|a|<1$ singularity is <u>inside</u> curve &

$$\int_c = \sqrt{\frac{\pi}{2}}\, x^{-1/2}\left[\frac{(1+i)}{(1-a)}\, e^{-i\,2/3\,x} - \frac{(1-i)}{(1+a)}\, e^{+i\,2/3\,x}\right]$$

$$-2\pi i\, e^{iax(a^2/3-1)}$$

(ii) if $a=\pm 1$ use Cauchy P.V. for singularity on curve.

$$\int_c = -\pi i\, e^{iax(a^2/3-1)}$$

$$\mp \frac{(1\mp i)}{(1\pm a)}\, e^{\pm i\,2/3\,x}\sqrt{\frac{\pi}{2}}\, x^{-1/2}$$

according to whether $a = \pm 1$

$$= \begin{cases} -\pi i\, e^{-i\,2/3\,x} - \frac{(1-i)}{2}\sqrt{\frac{\pi}{2x}}\, e^{i\,2/3\,x}, & a=+1 \\ \left(-\pi i + \frac{(1+i)}{2}\sqrt{\frac{\pi}{2x}}\right) e^{-i\,2/3\,x}, & a=-1 \end{cases}$$

5. $I(r) = \int_0^\infty \frac{r\,x\,dx}{(r^2+x)^{3/2}(1+x)}$, as (i) $r \to 0$
(ii) $r \to \infty$

(i) $r \to 0$

If $x = O(r^2)$ $\quad I \sim \frac{r \cdot r^2 \cdot r^2}{r^3} = O(r^2)$

If $x = O(1)$ $\quad I \sim \frac{r}{1} \cdot 1 = O(r)$ → leading order is global

$$\int_{'0'}^{\infty} \frac{r x}{x^{3/2}(1+x)}\,dx = \pi r \qquad (\text{indeed } O(r))$$

→ do not use this limit

Next term comes from region $O(r^2)$ (subtract off global contribution & scale $x = r^2 \xi$

$$\to r^3 \int_0^{'\infty'} \left[\frac{r^2 \xi}{r^3(1+\xi)^{3/2}} - \frac{1}{r\xi^{1/2}}\right] \frac{1}{1+r^2\xi}\,d\xi$$

→ take leading order term.

$$\to r^2 \int_0^{'\infty'} \left[\frac{\xi}{(1+\xi)^{3/2}} - \frac{1}{\xi^{1/2}}\right] d\xi$$

$$\to -r^2 \left[\frac{2}{\sqrt{1+\xi}} + 2\sqrt{1+\xi} - 2\sqrt{\xi}\right]\Bigg|_{\xi=0} = -4r^2$$

∴ leading 2 terms are:

$$\boxed{I(x) \sim \pi r - 4r^2 \quad \text{as } r \to 0}$$

5.
(cont.)

(ii)

$$I(r) = \int_0^\infty \frac{r x \quad dx}{(1+x)(r^2+x)^{3/2}} \qquad \text{let } \varepsilon = 1/r \quad , \quad \begin{matrix} r \to \infty \\ \varepsilon \to 0 \end{matrix}$$

$$\int_0^\infty \frac{\varepsilon^{-1} x \, dx}{(1+x)\,(\cancel{\varepsilon^{-2}}(1+\varepsilon^2 x))^{3/2}} \qquad \frac{\varepsilon^{-1}}{\varepsilon^{-3}} \qquad \varepsilon^2$$

$$\to \varepsilon^2 \int_0^\infty \frac{x\, dx}{(1+x)(1+\varepsilon^2 x)^{3/2}} \qquad \text{let } x = \frac{1}{\mu}, \quad dx = -\frac{1}{\mu^2} d\mu$$

$$\varepsilon^2 \int_0^\infty \frac{\cancel{\mu^{-1}}\, \mu^{-2} \quad d\mu}{\cancel{\mu^{-1}}(\mu+1)\,(\mu+\varepsilon^2)^{3/2}\, \mu^{-3/2}} \qquad \frac{-3/2}{(\varepsilon^2+\mu)^{5/2}} \qquad \varepsilon^2$$

$$\boxed{I = \underset{= \frac{1}{r^2}}{\varepsilon^2} \int_0^\infty \frac{d\mu}{\mu^{1/2}(1+\mu)(\varepsilon^2+\mu)^{3/2}}}$$

choose $\boxed{\varepsilon^2 \ll \delta \ll 1}$ $\quad \underset{I_1}{\varepsilon^2 \int_0^\delta} + \underset{I_2}{\varepsilon^2 \int_\delta^\infty}$

~~drop ε^2 factor for now~~

I_1 : scale $\mu = \varepsilon^2 \alpha$

$$I_1 = \varepsilon^2 \int_0^{\delta/\varepsilon^2} \frac{\varepsilon^2\, d\mu}{\varepsilon\, \alpha^{1/2}\, (1+\varepsilon^2\alpha)\, \varepsilon^3 (1+\alpha)}$$

$$= \cancel{\varepsilon^2} \int_0^{\delta/\varepsilon^2} \frac{d\alpha}{\alpha^{1/2}(1+\alpha)^{3/2}(1+\varepsilon^2\alpha)}$$

$$\boxed{I_1 = \cancel{\varepsilon^2} \int_0^{\delta/\varepsilon^2} \frac{1}{\alpha^{1/2}(1+\alpha)^{3/2}} \left[\underset{I_{1a}}{1} \underset{I_{1b}}{- \varepsilon^2 \alpha} \; \underset{I_{1c}}{+ \varepsilon^4 \alpha^2} \right] d\alpha} \qquad (\delta' = \delta/\varepsilon^2 \text{ large})$$

$$I_2 = \varepsilon^2 \int_\delta^\infty \frac{d\mu}{\mu^{\frac{1}{2}}(1+\mu)(\varepsilon^2+\mu)^{3/2}} \qquad \mu \geq \delta \gg \varepsilon^2$$

$$I_2 = \varepsilon^2 \int_\delta^\infty \frac{d\mu}{\mu^{1/2}(1+\mu)} \left[\frac{1}{\mu^{3/2}} - \frac{3}{2}\frac{\varepsilon^2}{\mu^{5/2}} \right] = \varepsilon^2 \int_\delta^\infty \frac{d\mu}{\mu^2(1+\mu)} \left[1 - \frac{3}{2}\frac{\varepsilon^2}{\mu} \right]$$

$$\boxed{I_2 = \varepsilon^2 \int_\delta^\infty \frac{d\mu}{\mu^2(1+\mu)} \left[1 - \frac{3}{2}\frac{\varepsilon^2}{\mu} \right]} \quad (\delta \text{ small})$$

I_{2a} $\quad$ I_{2b}

5. (cont.)

$$-\varepsilon^2 \int_0^{\delta/\varepsilon^2} \frac{\alpha^{1/2}}{(1+\alpha)^{3/2}}\, d\alpha \quad , \quad \text{let } \eta = \alpha^{1/2}, \quad \eta^2 = \alpha, \quad 2\eta\, d\eta = d\alpha$$

$$-\varepsilon^2 \int_0^{(\delta/\varepsilon^2)^{\frac{1}{2}}} \frac{\eta\, 2\eta\, d\eta}{(1+\eta^2)^{3/2}} \to \boxed{-2\varepsilon^2 \int_0^{(\delta/\varepsilon^2)^{1/2}} \frac{\eta^2\, d\eta}{(1+\eta^2)^{3/2}}} \quad I_{1b}$$

$$\text{or} \quad \boxed{-2\varepsilon^2 \int_0^{\delta'} \frac{\eta^2\, d\eta}{(1+\eta^2)^{3/2}} \quad , \quad \delta' \gg 1 . \quad \left(\delta' = \left(\frac{\delta}{\varepsilon^2}\right)^{1/2}\right)} \quad I_{1b}$$

$$\boxed{I_{1a} = \frac{2\sqrt{\delta'}}{\sqrt{\delta'+1}} \sim 2 - \frac{\varepsilon^2}{\delta} + \cancel{\frac{3}{4}\frac{\varepsilon^4}{\delta^2}} \text{ ignore}}$$

$$I_{1b} = \frac{+2\varepsilon^2}{\cancel{\sqrt{\ldots}}}\left[\frac{\delta'}{\sqrt{1+\delta'^2}} + \ln\left(-\delta' + \sqrt{\delta'^2+1}\right)\right]$$

$$\sim +2\varepsilon^2\left[+1 - \ln 2 - \ln\delta' - \frac{3}{4}\frac{1}{\delta'^2}\right]$$

$$\sim 2\varepsilon^2\left[1 - \ln 2 - \frac{1}{2}\ln\delta + \ln\varepsilon - \frac{3}{4}\frac{\varepsilon^2}{\delta}\right]$$

$$\boxed{I_{1b} \sim 2\varepsilon^2 - 2\varepsilon^2\ln 2 - \varepsilon^2\ln\delta - 2\varepsilon^2\ln\varepsilon - \cancel{\frac{3}{2}\frac{\varepsilon^4}{\delta}} \text{ ignore}}$$

$$\boxed{I_2 = \varepsilon^2\left[\frac{1}{\delta} + \ln\delta - \ln(1+\delta)\right] \sim \frac{\varepsilon^2}{\delta} + \varepsilon^2\ln\delta - \cancel{\varepsilon^2\delta} \text{ ignore}}$$

$$\cancel{\to I \sim 2 + \frac{2}{r^2}(1-\ln 2) - \frac{2\ln r}{r^2}}$$

I_{2b} : $\sim -\frac{3}{2}\varepsilon^4/\delta^2 + \frac{3}{2}\varepsilon^4/\delta$ $+ \cancel{\ldots}$

I_{1c} : $\sim \varepsilon^4\delta^2 + \left(\frac{5}{2} - 3\ln 2\right)\varepsilon^4 - \frac{15}{8}\varepsilon^4/\delta^2$ (δ terms will cancel)

5.
(cont.)

Add contributions:

$$\boxed{I \sim 2 + \frac{2}{r^2}\left(1 - \ln 2\right) - \frac{2\ln r}{r^2} + \frac{5/2 - 3\ln 2}{r^4}}$$

(putting ~~εa~~ $\varepsilon = \frac{1}{r}$)