Assignment 1 – Math 744

1. (a) Consider integral $-\int_x^\infty \frac{e^{it}}{t}\,dt$ and take real part.

Int. by parts: $u = -\frac{1}{t}$, $du = \frac{1}{t^2}\,dt$
$~~du~~\; dV = e^{it}\,du$, $V = -ie^{it}$

$$\to I = \frac{i}{t}e^{it}\Big|_x^\infty + \int_x^\infty \frac{ie^{it}}{t^2}\,dt, \quad u = \frac{i}{t^2}, \quad du = -\frac{2i\,dt}{t^3}$$

$$\to I = -\frac{ie^{ix}}{x} - \frac{e^{ix}}{x^2} + \int_x^\infty \frac{2}{t^3}e^{it}\,dt, \quad u = \frac{2}{t^3} \quad du = -\frac{6}{t^4}dt$$

$$I = -\frac{ie^{ix}}{x} - \frac{e^{ix}}{x^2} + \frac{2i}{x^3}e^{ix} - \int_x^\infty \frac{6i}{t^4}\,e^{it}\,dt, \quad u = \frac{3\cdot 2\cdot i}{t^4}$$

$$du = \frac{-4\cdot 3\cdot 2\cdot i\,dt}{t^5}$$

$$I = -\frac{ie^{ix}}{x} - \frac{e^{ix}}{x^2} + \frac{2ie^{ix}}{x^3} + \frac{3!\,e^{ix}}{x^4} - \int_x^\infty \frac{4!\,e^{it}}{t^5}\,dt$$

Take real part:

$$\therefore Ci(x) = \frac{\sin x}{x} - \frac{\cos x}{x^2} - \frac{2\sin x}{x^3} + \frac{3!\cos x}{x^4} + \underbrace{4!\int_x^\infty \frac{\cos t}{t^5}\,dt}_{R}$$

(b) Need to show that $R = o\left(\frac{1}{x^4}\right)$, i.e. that R decreases faster than $1/x^4$ as $x \to +\infty$ (~~assume~~, since $|\cos x| \le 1$ don't need to consider $\cos x$ part provided $\cos x \neq 0$, and in this case → previous term, i.e. $-\sin x$ term.

Now, $t \geq x$ so $|R| \leq \frac{4!}{x^5}\left|\int_x^\infty \cos t\,dt\right| = \frac{4!}{x^5}\underbrace{\left|\lim_{L\to\infty} \sin L - \sin x\right|}_{\leq 2}$

$$\to |R| \leq \frac{2\cdot 4!}{x^5} = o\left(\frac{1}{x^4}\right) \checkmark$$

(N.B. $\sin x$, $\cos x$ are not asymptotic expansion arguments since they can be factored out. Actual asymptotic sequence are inverse powers of x: $\frac{1}{x^n}$. $Ci(x) \sim \frac{\sin x}{x}\left(1! - \frac{2!}{x^2} + \frac{3!}{x^4} + \dots\right) + \frac{\cos x}{x}\left(\frac{1!}{x} - \frac{3!}{x^3} + \dots\right)$

(b) (cont.) The asymptotic series does not converge. The asymptotic approximation is valid when $x \to \infty$ with N (number of terms) fixed

(c) Best result is obtained by retaining terms until the next term is larger:

N	|term with $x = 2.5$|
1	0.2394
2	0.1282
3	0.0766
4	0.1231

increasing → $N = 3$ terms is optimal

$$\rightarrow \quad Ci(2.5) = 0.2394 + 0.1282 \mp 0.0766 = \boxed{0.2910}$$

(exact value is $0.2858 \rightarrow \sim 2\%$ error!)

2.(a)

$$I = \int_x^{\infty} u^{-p} e^{-u}\, du = \int_x^{\infty} t^{-p} e^{-t}\, dt,$$

let $u = t^{-p}$ $\quad dv = e^{-t}\, dt$

$du = -p\, t^{-p-1}\, dt \quad v = -e^{-t}$

$$\rightarrow I = \frac{e^{-x}}{x^p} - \int_x^{\infty} p\, t^{-p-1} e^{-t}\, dt, \quad u = p\, t^{-p-1}, \; du = -p(p+1)\, t^{-p-2}\, dt$$

$$= \frac{e^{-x}}{x^p} - \frac{p\, e^{-x}}{x^{p+1}} + \int_x^{\infty} p(p+1)\, t^{-p-2} e^{-t}\, dt, \quad u = p(p+1)\, t^{-p-2}, \; du = -p(p+1)(p+2)\, t^{-1}\, dt$$

$$= \frac{e^{-x}}{x^p} - \frac{p\, e^{-x}}{x^{p+1}} + \frac{p(p+1)\, e^{-x}}{x^{p+2}} - \int_x^{\infty} p(p+1)(p+2) \frac{e^{-t}}{t^{p+3}}\, dt$$

(a) $I = e^{-x}\left(\frac{1}{x^{p}} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}}\right) - p(p+1)(p+2)\int_x^{\infty} e^{-t}\, t^{-p-3}\, d$ $\square$

(b) Need to show $e^{x}\left|\int_x^{\infty} e^{-t}\, t^{-p-3}\, dt\right| = o\left(\frac{1}{x^{p+2}}\right)$

Now, $t \geq x \rightarrow t^{-p-3} \leq x^{-p-3}$

$$\rightarrow e^{x}\left|\int_x^{\infty} e^{-t}\, t^{-p-3}\, dt\right| \leq \frac{e^{x}}{x^{p+3}}\left|\int_x^{\infty} e^{-t}\, dt\right|$$

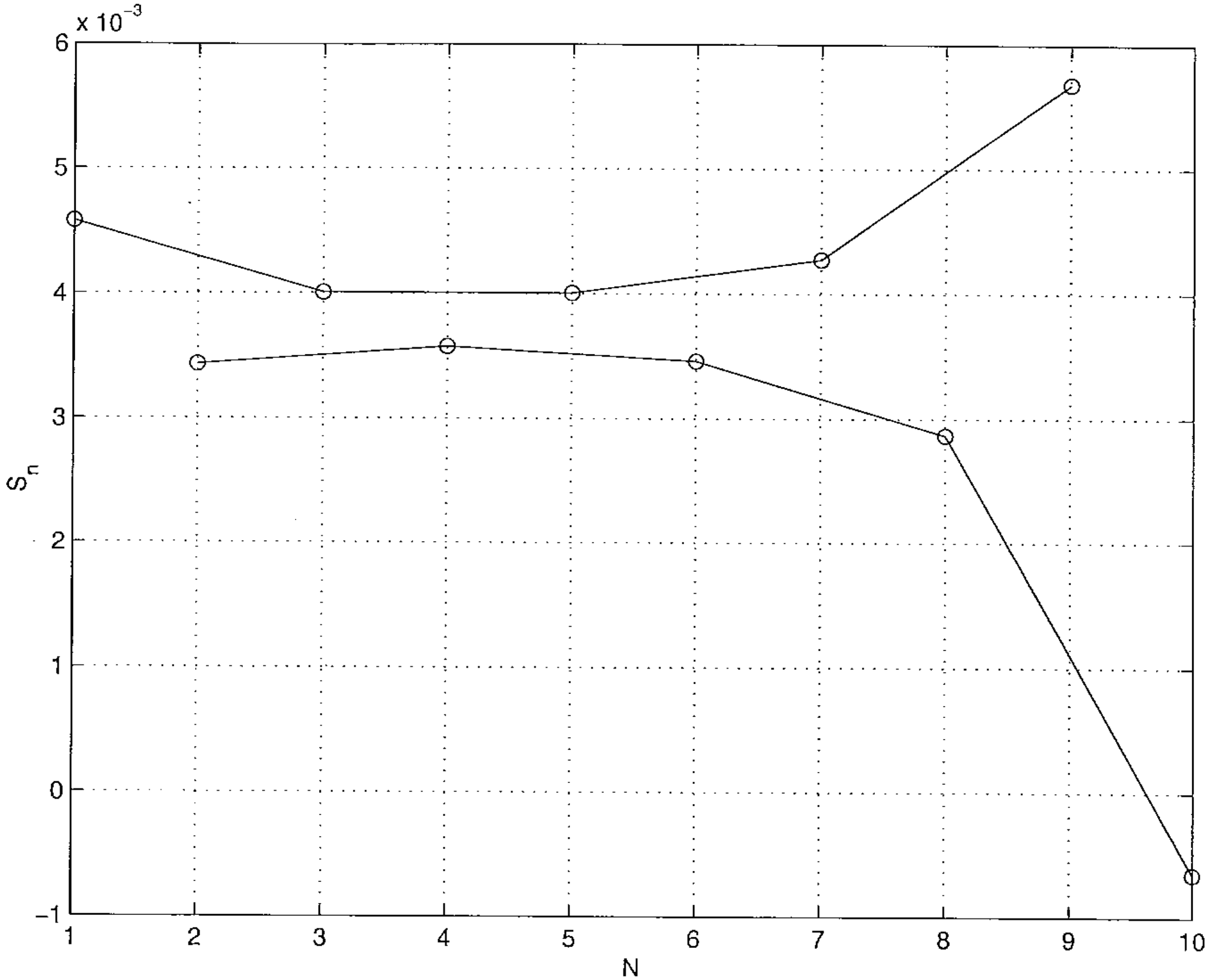
$$= \frac{e^{x}}{x^{p+3}}\, e^{-x} = \frac{1}{x^{p+3}} = o\left(\frac{1}{x^{p+2}}\right.$$

This series does <u>not</u> converge (property of asymp. series!).

(c) see attached plot.

r t Q.2(c)

Partial Sums for $I(4,1)$



- optimal number of terms is ~ 4 or 5
- estimated value is

$3.5773\times10^{-3} \leq I(4,1) \leq \cancel{4\times10^{-3}}\ 4.0065\times10^{-3}$

→ agrees with "exact" value $I(4,1) \approx 3.779\times10^{-3}$

3. $$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{\nu t - z\cosh t}\, dt \quad \text{as } \nu \to \infty,\ z = O(1)$$

max. of exponent at $t_c = \operatorname{arcsinh}\left(\frac{\nu}{z}\right)$

For large ν, $z = O(1)$

$$\boxed{t_c \sim \ln\left(\frac{2\nu}{z}\right)}$$

$\Big[$ since $\frac{1}{2}\left(e^{t_c} - \cancel{e^{-t_c}}\right) = \frac{\nu}{z} \gg 1$
$\Rightarrow e^{t_c} = \frac{2\nu}{z} \Rightarrow t_c = \log\left(\frac{2\nu}{z}\right)$

Expand exponent in Taylor series about $t = t_c$

$$\Rightarrow \nu t_c - z\cosh t_c - \frac{1}{2} z\cosh t_c\, (t - t_c)^2$$

$$\Rightarrow K_\nu(z) \sim \frac{1}{2}\exp\left[\nu t_c - z\cosh t_c\right] \int_{-\infty}^{\infty} e^{-\frac{1}{2} z\cosh t_c\, (t-t_c)^2}\, dt$$

width of significant region is:

$$[z\cosh t_c]^{-1/2} = \left[\frac{z}{2}\left(e^{t_c} + e^{-t_c}\right)\right]^{-1/2}$$

$$= \left[\frac{z}{2}\left(\frac{2\nu}{z} - \underbrace{\frac{z}{2\nu}}_{\text{neg.}}\right)\right]^{-1/2} \sim \nu^{-1/2} \text{ small as } \nu \to \infty$$

OK. (no need for re-scaling)

$$\Rightarrow K_\nu(z) \sim \frac{1}{2}\left(\frac{2\nu}{z}\right)^{\nu} e^{-\nu} \int_{-\infty}^{\infty} e^{-\frac{1}{2} z\cosh t_c\, (t-t_c)^2}\, dt$$

$\Rightarrow$ let $\tau = (t - t_c)\sqrt{z\cosh t_c}$

$$d\tau = \sqrt{z\cosh t_c}\, dt \sim \sqrt{\nu}\, dt$$

$$\Rightarrow K_\nu(z) \sim \frac{1}{2}\left(\frac{2\nu}{z}\right)^{\nu} \sqrt{\frac{2\pi}{\nu}}\, e^{-\nu}$$

or $$\boxed{K_\nu(z) \sim \sqrt{\frac{\pi}{2\nu}}\left(\frac{2\nu}{z}\right)^{\nu} e^{-\nu} \quad \text{as } \nu \to \infty,\ z = O(1)}$$

4. $$I(x) = \int_0^\infty \exp(-xt - (\ln t)^2)\, t^{-1}\, dt, \quad x \to \infty$$
$$= \int_0^\infty \exp(-xt - (\ln t)^2 - \ln t)\, dt$$

where is max. pt?

solution of $\quad -x - 2\frac{\ln t}{t} - \frac{1}{t} = 0$

or $\quad xt + 2\ln t + 1 = 0 \qquad f(x,t) = 0$

transcendental eqn $\rightarrow$ try Newton's method

initial guess $t_c = 1/x$ (roughly correct)

$\cancel{x_1 = x_0 - \partial f/\partial x \; f(x_1}$

$$t_1 = t_0 - \frac{f(x,t_0)}{f'(x,t_0)}$$

$$f(x,t_0) = xt_0 + 2\ln t_0 + 1 = 2(1 - \ln x)$$

$$\frac{\partial f}{\partial t}(x,t_0) = x + 2/t_0 = 3x$$

$$\therefore t_1 = \frac{1}{x} - \frac{2(1 - \ln x)}{3x} = \frac{1 + 2\ln x}{3x}$$

$$\therefore \boxed{t_c \approx \frac{1+2\ln x}{3x}}\boxed{\approx \frac{2}{3}\frac{\ln x}{x}}, \quad \cancel{\text{2 iterations} \rightarrow t_c \approx 4/5 \frac{\ln x}{x}}$$

Now shift max to zero: $t = t_c + \tau$

$$\rightarrow \int_{-\infty}^{\infty} \exp\left(-x(t_c+\tau) - [\ln(t_c+\tau)]^2 - \ln(t_c+\tau)\right) d\tau$$

& expand in Taylor series for small τ: (Noting that $O(\tau) \approx 0$)

$$\rightarrow \quad e^{-xt_c - (\ln t_c)^2 - \ln t_c} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\frac{1}{t_c^2}(2\ln t_c - 1)\,\tau^2\right] d\tau$$

(note that width of exp gaussian is small as $x \to \infty$ so there is no need for re-scaling)

$$\therefore \boxed{I(x) \sim \frac{\sqrt{2\pi}}{\sqrt{|\ln t_c^2 - 1|}} \exp\left[-xt_c - (\ln t_c)^2\right], \quad x \to \infty} \quad (*)$$

where $t_c \approx \dfrac{\ln x^{2/3}}{x}$

Note: for more accurate results (esp. for large x), use maple solution for t_c

$$t_c = \frac{2}{x}\,\mathrm{LambertW}\left(\frac{1}{2} x e^{-1/2}\right) \text{ in } (*)$$

where LambertW is solution to the equation

$$y e^{y} = x .$$