Assignment 1 **MATH 744** **23 September 2010**

This assignment is due 7 October.

1. Consider the cosine integral function

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,\mathrm{d}t.$$

[6] (a) Using integration by parts repeatedly on the integral

$$-\int_x^\infty \frac{e^{it}}{t}\,\mathrm{d}t,$$

show that the asymptotic expansion for $\mathrm{Ci}(x)$ is given by

$$\mathrm{Ci}(x) = \frac{\sin x}{x} - \frac{\cos x}{x^2} - \frac{1\cdot 2\ \sin x}{x^3} + \frac{1\cdot 2\cdot 3\ \cos x}{x^4} + R,$$

where the remainder is

$$R = -4!\int_x^\infty \frac{\cos t}{t^5}\,\mathrm{d}t.$$

[4] (b) Show that the series you found in (a) is indeed asymptotic. Does the asymptotic expansion you found in (a) converge? Under what conditions should it be a valid approximation to $\mathrm{Ci}(x)$?

[4] (c) Use the asymptotic expansion to give the best estimate of the value of $\mathrm{Ci}(2.5)$ that you can (use radians). Justify your answer.

[8] 2. (a) Use integration by parts repeatedly to show that the asymptotic expansion of the generalized incomplete gamma function $I(x,p)$ for large x is given by

$$I(x,p) = \int_x^\infty u^{-p}e^{-u}\,\mathrm{d}u = e^{-x}\left(\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}}\right) - p(p+1)(p+2)\int_x^\infty e^{-u}u^{-p-3}\,\mathrm{d}u \qquad (1)$$

[6] (b) Show that the series you found in (a) is an asymptotic series. Does the series converge?

[8] (c) If the complete series has the form

$$\int_x^\infty u^{-p} e^{-u}\,\mathrm{d}u \sim e^{-x} \sum_{n=1}^{N} (-1)^{n-1} \frac{(p+n-2)!}{(p-1)!x^{p+n-1}}.$$

Plot the the first 10 partial sums for $p = 1$, $x = 4$ as a function of N (you may use a computer, e.g. matlab or maple!). From your graph, what is the optimal number of terms to take? What is the estimated range for the value of $I(4, 1)$? Does this estimate agree with the 'exact' value of $I(4, 1) = 3.779352410 \times 10^{-3}$?

[6] 3. Find the asymptotic behaviour of

$$K_\nu(z) = \frac{1}{2} \int_{-\infty}^{\infty} e^{\nu t - z \cosh t},\ \mathrm{d}t$$

for real ν and z and with $z = O(1)$ in the limit $\nu \to \infty$.

[8] 4. Find the leading behaviour of

$$I(x) = \int_0^\infty e^{-xt-(\ln t)^2} t^{-1}\,\mathrm{d}t \quad \text{as} \quad x \to \infty.$$

Hint: use one iteration of Newton's method with a reasonable initial guess to estimate the position of the maximum of the integrand.

[Total: 50]