MATHEMATICS 3C

15 NOVEMBER 2005 — 2ND MIDTERM EXAM — DR. N. KEVLAHAN

Duration of Exam: 50 minutes — Please show all work and calculations

1. Consider the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}.$$

[4] (a) Find the eigenvalues and eigenvectors of A.

[1] (b) Why is A diagonalizable?

[4] (c) Find the matrices S and S^{-1} such that $S^{-1}AS$ is diagonal.

[4] (d) Find e^{At}, where t is a real variable.

[2] (e) Use the above results to solve the following system of linear differential equations

$$\begin{aligned} \dot{x}_1(t) &= 2x_1(t) + 2x_2(t) \\ \dot{x}_2(t) &= x_1(t) + 3x_2(t) \end{aligned}$$

with initial conditions $[x_1(0), x_2(0)]^T = [1, 1]^T$.

2. Consider the differential equation

$$4xy'' + 2y' + y = 0,$$

where $y = y(x)$.

[1] (a) How many linearly independent solutions does this differential equation have?

[3] (b) Find and classify all singular points of this equation.

[6] (c) Find the roots of the indicial equation for the Frobenius solution of the differential equation about $x = 0$. How many linearly independent solutions will the Frobenius method produce (and why)?

[4] (d) Find the recurrence relation associated with the larger root (or the unique root if it is double).

[1] (e) What is the radius of convergence of the solution associated with the larger root (or the unique root if it is double)?

Total: 30

THE END

Math 3C - Test 2 Solutions

1. (a) $$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\rightarrow (2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = 0 \rightarrow (\lambda-1)(\lambda-4)$$

$$\therefore \boxed{\lambda_1 = 1, \ \lambda_2 = 4}$$

$$\lambda_1 = 1: \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \vec{\xi}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 4: \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \vec{\xi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) A is diagonalizable because it is (2×2) and has <u>2 distinct e-values</u> (and hence 2 L.I. e-vectors).

(c) $$S = \left[\vec{\xi}_1 \ \vec{\xi}_2\right] = \boxed{\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}}$$

Find inverse: $$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 3 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 3 & 0 & 1 & -1 \\ 0 & 3 & 1 & 2 \end{array}\right] \rightarrow \boxed{S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \frac{1}{3}}$$

(d) $e^{At} = S e^{Dt} S^{-1}$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{4t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{t} & -e^{t} \\ e^{4t} & 2e^{4t} \end{bmatrix} = \boxed{\frac{1}{3}\begin{bmatrix} 2e^{t}+e^{4t} & -2e^{t}+2e^{4t} \\ -e^{t}+e^{4t} & e^{t}+2e^{4t} \end{bmatrix}}$$

(e) solution is $\vec{x} = e^{At}\vec{x}_0$:

$$= \frac{1}{3}\begin{bmatrix} 3e^{4t} \\ 3e^{4t} \end{bmatrix} = \boxed{e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{x}(t)}$$

2. (a) A 2nd order linear ODE has 2 L.I. solutions

(b) Put in std form:

$$\boxed{y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0}$$

only $\boxed{\text{S.P. at } x = 0}$

$\boxed{\text{regular}}$ since $\lim\limits_{x\to 0} \frac{x}{2x} = \frac{1}{2} < \infty$

$$\lim_{x\to 0} \frac{x^2}{4x} = 0 < \infty.$$

(c) let $y = \sum\limits_{k=0}^{\infty} a_k x^{k+r}$, $y' = \sum\limits_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$

$$y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2}$$

sub in eqn:

$$4\sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1} + 2\sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

$$+ \sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

lowest power: x^{r-1}

coefficients: $4a_0 r(r-1) + 2a_0 r = 0 \quad (a_0 \neq 0)$

$$\to 2r(2r-2+1) = 0$$

$$\to r(2r-1) = 0 \to \boxed{r_1 = \tfrac{1}{2},\ r_2 = 0}$$

since roots are distinct and do not differ by an integer we will obtain two solutions in Frobenius form.

(d) set $r = 1/2$

$$\Rightarrow 4\sum_{k=0}^{\infty} a_k (k+1/2)(k-1/2)\, x^{k-1/2} + 2\sum_{k=0}^{\infty} a_k (k+1/2)\, x^{k-1/2}$$

$$+ \sum_{k=0}^{\infty} a_k\, x^{k+1/2} = 0$$ (shift indices $k'-1/2 = k+1/2$ $\Rightarrow k = k'-1$)

$$\Rightarrow a_k\, 2(k+1/2)\,[2(k-1/2)+1] + a_{k-1} = 0$$

$$\Rightarrow \boxed{a_k = -\frac{a_{k-1}}{(2k+1)\,2k}}$$ "recurrence relation"

(e) Radius of convergence is $\boxed{\infty}$ (distance to nearest singularity).