MATHEMATICS 3C

11 OCTOBER 2005 — 1ST MIDTERM EXAM — DR. N. KEVLAHAN

Duration of Exam: 50 minutes — Please show all work and calculations

[2] 1. (a) How many linearly independent eigenvectors does an $(n \times n)$ self-adjoint matrix have?

[2] (b) Show that if A is orthogonal then $\det(A) = \pm 1$. [Hint: consider $\det(AA^T)$.]

[4] (c) Show that the eigenvalues of a unitary matrix U have absolute value 1. [Hint: consider $\langle Ux, Ux \rangle$.]

[4] 2. (a) Show that $\{1, t-1\}$ is a basis for the vector space P^1 (polynomials of degree ≤ 1).

[2] (b) Show that the function $\mathcal{L}(f) = 2df/dt$ is a linear operator.

[4] (c) Find the matrix of the linear operator in (b) with respect to the basis in (a).

[2] (d) Use the matrix from (c) to apply the linear operator to the vector $3 + 2t$.

[4] 3. (a) When is the general (2×2) matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible?

[2] (b) Consider the following linear system:

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Does it have a unique solution (why or why not)?

[4] (c) Consider the following three vectors in R^3,

$$\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}.$$

Are these vectors linearly independent (why or why not)? [Hint: use the determinant.]

Total: 30

THE END

Math 3C test 1

1. (a) An $(n \times n)$ S.A. matrix has $\boxed{n}$ L.I. e-vectors

(b) Orthogonal: $A^T = A^{-1}$

$$\det(AA^T) = \det(AA^{-1}) = \det(I) = 1$$

$$\det(A)\det(A^T) = \det(A)^2$$

$$\therefore \det(A)^2 = 1 \longrightarrow \det(A) = \pm 1 \quad \square$$

(c) $\langle Ux, Ux \rangle = \langle x, x \rangle$ (U is unitary)

if x is an e-vector w with e-value λ

$$\langle \lambda x, \lambda x \rangle = \lambda^* \lambda \langle x, x \rangle = |\lambda|^2 \langle x, x \rangle$$

$$\rightarrow |\lambda|^2 \langle x, x \rangle = \langle x, x \rangle \Longrightarrow |\lambda| = 1 \quad \square$$

2. (a) A basis is a <u>L.I.</u> set of vectors that <u>spans</u> the v.S.

* $\{1, t-1\}$ clearly <u>spans</u> P^1. ✓

* check L.I. : $\lambda_1 \cdot 1 + \lambda_2 (t-1) = 0$

$\rightarrow \lambda_1 - \lambda_2 = 0$ (coeff. 1)

$\lambda_2 = 0$ (coeff. t)

$\rightarrow$ only solution is $\lambda_1 = \lambda_2 = 0 \Rightarrow$ L.I. □ ✓

(b) Need to check commutation with L.C. :

$L(u+v) = 2(u+v)' = 2u' + 2v' = L(u) + L(v)$ ✓

$L(\lambda u) = 2(\lambda u)' = 2\lambda u' = \lambda 2u' = \lambda L(u)$ ✓

(c) $L(1) = 0$

$L(t-1) = 2$

$\rightarrow \begin{bmatrix} 0 & +2 \\ 0 & 0 \end{bmatrix}$

(d) $3 + 2t = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ ✓

3. (a) For a unique solution $\det(A) \neq 0$

$\therefore$ Need $ad \neq bc$ (ie. $ad - bc \neq 0$).

(b) $\boxed{\text{No!}}$ Since $\det\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} = 3\cdot 4 - 2\cdot 6 = 0$.

(c) Test for L.I. in $\mathbb{R}^n$: Form matrix ~~of cols~~ whose cols are vectors. If ~~no~~ inverse exists $\Rightarrow$ unique soln exists.

$$\rightarrow \begin{bmatrix} 7 & -2 & 5 \\ 0 & 4 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

$\det = 7\cdot 4\cdot(-1) = -28$
$\neq 0$
$\therefore$ vectors are L.I.