Math 3C Assignment 6 Solutions (1)

1. $\kappa\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right) = \dfrac{\partial u}{\partial t}$, $0<r<1$

$u(1,t) = 0$ (BC), $u(r,0) = f(r)$ (IC)

(a) let $u(r,t) = F(r)G(t)$

$\Rightarrow F''G + \frac{1}{r}F'G = \frac{1}{\kappa}FG' \quad \times \frac{1}{FG}$

$\Rightarrow \frac{F''}{F} + \frac{1}{r}\frac{F'}{F} = \frac{1}{\kappa}\frac{G'}{G} = k$ (const.)

$$\Rightarrow \boxed{\begin{array}{ll} F'' + \frac{1}{r}F' - kF = 0 & (1) \\ G' = \cancel{k}\kappa G & (2) \end{array}}$$

(b) expect temperature to decrease $\rightarrow \boxed{k = -\nu^2 < 0}$

(2) $\rightarrow \boxed{G(t) = c_1 \exp(-\nu^2 \kappa t)}$

(1) $\rightarrow F'' + \frac{1}{r}F' + \nu^2 F = 0$ zeroth-order Bessel eqn.

let $\boxed{\rho = \nu r}$ $\quad \frac{\partial}{\partial r} = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial r} = \nu\frac{\partial}{\partial \rho}$, $\frac{\partial^2}{\partial r^2} = \nu^2\frac{\partial^2}{\partial \rho^2}$

std form $\rightarrow \nu^2 F_{\rho\rho} + \frac{\nu^2}{\rho}F_\rho + \nu^2 F = 0$

$$\rightarrow \boxed{F_{\rho\rho} + \frac{1}{\rho}F_\rho + F = 0}$$

solution: $F(\rho) = d_1 J_0(\rho) + \cancel{d_2 N_0(\rho)}$ blows up at $\rho = 0$

To satisfy B.C. $J_0(\nu \cdot 1) = 0$

$\Rightarrow \nu_m = \alpha_m^{(0)}$ (zeros of J_0)

$m = 1, 2, \cdots$

→ Normal modes are

$$\boxed{u_m(r,t) = J_0(\alpha_m^{(0)} r)\, \exp\left(-(\alpha_m^{(0)})^2 \kappa t\right)} \quad \boxed{2}$$

general solution is $\boxed{u(r,t) = \sum_{m=1}^{\infty} A_m u_m(r,t)}$ $\boxed{2}$

(c) Impose initial condition

$$f(r) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m^{(0)} r) \quad \boxed{2}$$

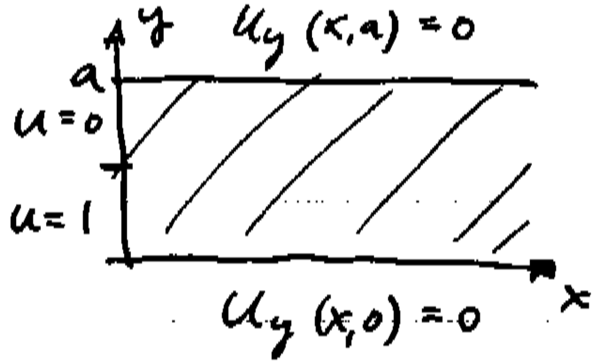
take inner product of both sides wrt $J_0(\alpha_n^{(0)} r)$

$$\Rightarrow \int_0^1 f(r)\, J_0(\alpha_n^{(0)} r)\, r\,dr = A_n \| J_0(\alpha_n^{(0)}) \|^2 \quad \boxed{2}$$

$$\Rightarrow \boxed{A_n = \frac{2}{[J_1(\alpha_n^{(0)})]^2} \int_0^1 f(r) J_0(\alpha_n^{(0)} r)\, r\,dr} \quad \boxed{2}$$

(remember to use correct weighted inner product!)

2.



u finite as $x \to \infty$

$\Delta u = 0$ let $u(x,y) = S(x)\,G(y)$

$$\rightarrow \quad \frac{S''}{S} = -\frac{G''}{G} = k \quad \text{(const.)}$$

ODE's → ① $S'' = kS$, S finite

② $G'' = -kG$, $G'(0) = G'(a) = 0$

solve ② → $k = \left(\frac{n\pi}{a}\right)^2$ to fit in with B.C.

$$G(y) = \cancel{A \sin \frac{n\pi}{a} y} + B \cos \frac{n\pi y}{a} \rightarrow \boxed{G(y) \propto \cos \frac{n\pi y}{a}}$$

① → $S(x) = A \sinh\left(\frac{n\pi x}{a}\right) + B \cosh\left(\frac{n\pi x}{a}\right)$

or (easier) $A' e^{-\frac{n\pi x}{a}} + \cancel{B' e^{+\frac{n\pi x}{a}}}$ need $S(x)$ finite as $x \to \infty$

$$\therefore \boxed{S(x) \propto e^{-\frac{n\pi x}{a}}}$$

Normal modes are $e^{-\frac{n\pi x}{a}} \cos \frac{n\pi y}{a}$

& general solution is

$$u(x,y) = \sum_{n=1}^{\infty} a_n e^{-\frac{n\pi x}{a}} \cos \frac{n\pi y}{a}$$

B.C. : $\boxed{a_n = \dfrac{\int_0^{a/2} \cos \frac{n\pi y}{a}\, dy}{\int_0^a \cos^2 \frac{n\pi y}{a}\, dy}}$ (note $n=0$ is a bit special!)

$$\rightarrow a_n = \begin{cases} \dfrac{a/2}{a} & \text{if } n = 0 \\[2ex] \dfrac{\dfrac{\cancel{a}}{n\pi}\sin\left(\dfrac{n\pi}{2}\right)}{\cancel{a}/2} \end{cases} = \begin{cases} 0 & n \text{ even} \\[1ex] \dfrac{2}{(2p-1)\pi}(-1)^{p-1} & n = 2p-1,\ n \text{ odd} \end{cases}$$

$\therefore$ solution is

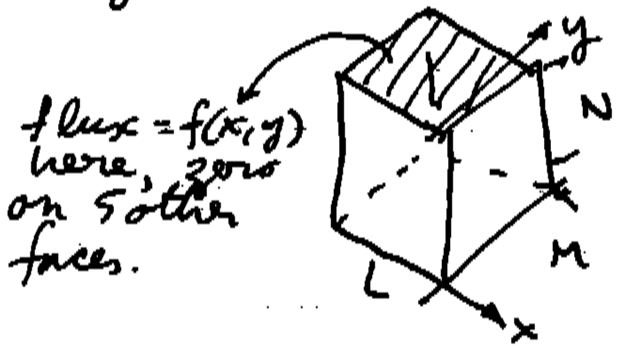
$$\boxed{u(x,y) = \frac{1}{2} + \frac{2}{\pi}\sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{(2p-1)} e^{-\frac{(2p-1)\pi x}{a}} \cos\left(\frac{(2p-1)\pi y}{a}\right)}$$

~~long time~~ large x solution is $u(x,y) \approx \frac{1}{2}$!

3. $\Delta u = 0$ ①

$0 = \frac{\partial u}{\partial x}(0,y,z) = \frac{\partial u}{\partial x}(L,y,z) = \frac{\partial u}{\partial y}(x,0,z) = \frac{\partial u}{\partial y}(x,M,z) = \frac{\partial u}{\partial z}(x,y,0$

$\frac{\partial u}{\partial z}(x,y,N) = f(x,y)$



(a) let $u(x,y,z) = X(x)Y(y)Z(z)$ + sub in ①

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = k \quad \text{(sep } x \text{ from } y + z\text{)}$$

→ ① $X'' = kX$

② $-\frac{Y''}{Y} - \frac{Z''}{Z} = k \to -\frac{Y''}{Y} = k + \frac{Z''}{Z} = l$

∴ ODE's with BC's are

$$\boxed{\begin{array}{lll} ① & X'' = kX, & X'(0) = X'(L) = 0 \\ ② & Y'' = -lY, & Y'(0) = Y'(M) = 0 \quad \text{not} \\ ③ & Z'' = (l-k)Z, & Z'(0) = 0,\ Z'(N) = \text{det by ODE} \end{array}}$$

(b) to fit in with zero flux boundary conditi[ons] we must have $k = -\left(\frac{n\pi}{L}\right)^2$, $\cancel{(\ell - k) = -\left(\frac{m\pi}{M}\right)^2}$ $\ell = \left(\frac{m\pi}{M}\right)^2$ to give cosine solutions:

1 : $X(x) \propto \cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2, \ldots$

2 : $Y(y) \propto \cos\left(\frac{m\pi y}{M}\right)$, $m = 0, 1, 2, \ldots$

3 $Z'' = \left[\left(\frac{m\pi}{M}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right] Z$, $Z'(0) = 0$

$$\rightarrow Z'' = \underbrace{\pi^2\left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2\right]}_{\lambda_{mn}^2 > 0 \rightarrow \sinh, \cosh} Z \quad , \quad ''$$

$$Z(z) = A \sinh(\lambda_{mn} z) + B \cosh(\lambda_{mn} z)$$

$$\rightarrow Z'(z) = A' \cosh(\lambda_{mn} z) + B' \sinh(\lambda_{mn} z)$$

$Z'(0) = 0 \rightarrow A' = 0$

$$\therefore Z(z) \propto \cosh(\lambda_{mn} z)$$

~~general solution~~ ~~Norm~~

$$\boxed{\begin{array}{l}\text{Normal modes: } u_{mn}(x,y,z) = \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{M}\right)\cosh(\lambda_{mn} z) \\ \text{where } \lambda_{mn} = \pi\left[\left(\frac{m}{M}\right)^2 + \left(\frac{n}{L}\right)^2\right]^{1/2}\end{array}}$$

&

$$\boxed{\begin{array}{l}\text{general solution is} \\ u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{M}\right)\cosh(\lambda_{mn} z)\end{array}}$$

(c) $\frac{\partial u}{\partial z}(x,y,N) = f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{nm} \overbrace{\cos\left(\frac{n\pi x}{L}\right)}^{X_n} \overbrace{\cos\left(\frac{m\pi y}{M}\right)}^{Y_m}$

$\times \quad \lambda_{mn}\sinh(\lambda_{mn}N)$

use orth. of X_n, Y_m

$\Rightarrow \langle\langle f, X_n\rangle, Y_m\rangle = c_{nm}\lambda_{mn}\sinh(\lambda_{mn}N) \times \|X_n\|^2 \|Y_m\|^2$

$$\therefore \boxed{c_{nm} = \frac{\langle\langle f, X_n\rangle, Y_m\rangle}{\lambda_{mn}\sinh(\lambda_{mn}N)\ \|X_n\|^2 \|Y_m\|^2}}$$

or in terms of integrals, using definition of inner product:

$$\boxed{c_{nm} = \frac{\int_0^M \int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{M}\right)dx\,dy}{\lambda_{mn}\sinh(\lambda_{mn}N)\int_0^L \cos^2\frac{n\pi x}{L}dx \int_0^M \cos^2\frac{m\pi y}{M}dy}}$$

(d) If $f(x,y) = \cos\left(\frac{5\pi x}{L}\right)\cos\left(\frac{3\pi y}{M}\right)$ only non-zero term is $\boxed{n=5,\ m=3}$!

$\Rightarrow \quad 1 = c_{53}\lambda_{35}\sinh(\lambda_{35}N)$

$$\therefore \boxed{c_{53} = \frac{1}{\lambda_{35}\sinh(\lambda_{35}N)}}$$

$$\& \boxed{u(x,y,z) = c_{53}\cos\left(\frac{5\pi x}{L}\right)\cos\left(\frac{3\pi y}{M}\right)\cosh(\lambda_{35}z)}$$

4.

Using fact that temperature does not depend on angle θ:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

Separation of variables: let $u(r,t) = F(R)T(t)$

$$\rightarrow F''T + \frac{1}{r}F'T = \frac{1}{c^2}FT' \quad \left(\times \frac{1}{FT}\right)$$

$$\rightarrow \frac{F''}{F} + \frac{1}{r}\frac{F'}{F} = \frac{1}{c^2}\frac{T'}{T} = -k^2$$ (temperature must decrease with time)

consider F eqn:

$F'' + \frac{1}{r}F' + k^2 F = 0$ let $r = \frac{x}{k}$

$\rightarrow F'' + \frac{1}{x}F' - F = 0$ this is Bessel eqn. with $n=0$

$\rightarrow F(x) = c_1 J_0(x) + c_2 \cancel{N_0(x)}$ → Blows up at $x=0$

B.C. $\frac{\partial u}{\partial r}(r=R)=0 \rightarrow F(R)\cancel{T(t)}=0 \rightarrow F(R)=0$

$\rightarrow J_0'(kR)=0$ ∴ $kR = \beta_m$ or $k = \frac{\beta_m}{R}$

where β_m are the extrema of J_0

$$\rightarrow \boxed{F_m(r) = c_1 J_0\left(\frac{\beta_m}{R} r\right)}$$

Now solve T eqn: $\frac{T'}{T} = -k^2c^2 = -\frac{c^2\beta_m^2}{R^2}$

$$\rightarrow \boxed{T_m(t) = c_2 \exp\left[-\frac{c^2\beta_m^2}{R^2}t\right]}$$

9

$\therefore$ general solution is ~~sum over~~ linear combination of all possible sol^{n}s m

$$u(r,t) = A_0 + \sum_{m=1}^{\infty} A_m J_0\left(\frac{\beta_m r}{R}\right) \exp\left[-\left(\frac{\beta_m c}{R}\right)^2 t\right]$$

But, as $t \to \infty$ sum $\to 0$ $\therefore A_0 = u_\infty$

and $$\boxed{u(r,t) = u_\infty + \sum_{m=1}^{\infty} A_m J_0\left(\frac{\beta_m r}{R}\right) \exp\left[-\left(\frac{\beta_m c}{R}\right)^2 t\right]}$$

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5. Laplace's eqn: $\Delta u = 0$ with B.C. $u(1,\theta) = \cos\theta$

write Δu in polar coordinates:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

separation of variables: $u(r,\theta) = F(r)\,T(\theta)$

$$\to F''T + \frac{1}{r}F'T + \frac{1}{r^2}FT'' = 0 \quad \times \frac{r^2}{FT}$$

$$\to r^2\frac{F''}{F} + r\frac{F'}{F} = -\frac{T''}{T} = k$$

For periodic solutions with periodic $\frac{2\pi}{n}$ need

$k = n^2$, $n = \cancel{0}, 1, 2, \ldots$

$\to T_n(\theta) = A_n \cos(n\theta + \delta_n)$, But B.C. $F(1)\,T(\theta) = \cos\theta$

$\to A_n = 1,\ n = 1,\ \delta_n = 0 \quad (k = 1)$

$\to \boxed{T(\theta) = T_1(\theta) = \cos\theta}$

F equation: $r^2F'' + rF' - 1 = 0$, let $F = r^\alpha$

$\to \alpha(\alpha - 1) + \alpha - 1 = 0 \Rightarrow \alpha = \pm 1$

or $F(r) = c_1 r + \cancel{c_2 r^{-1}} \to$ blows up at $r=0$

$\therefore$ solution is $\boxed{u(r,\theta) = F(r)\,T(\theta) = r\cos\theta}$

5 (cont.) Let $\xi = \dfrac{2mE}{\hbar^2} \rightarrow \Delta\psi = -\xi\psi$ (e-value problem!)

Let $\psi(x,y,z) = F(x,y)G(z)$ (separation of variables)

Then $(\Delta F)G + FG'' = -\xi FG \quad \times \dfrac{1}{FG}$

$\rightarrow \dfrac{\Delta F}{F} + \xi = -\dfrac{G''}{G} = k$

G satisfies $G'' + kG = 0$ with B.C. $G(0) = G(c) = 0$

$\therefore k = \mu_n^2$ with $\mu_n = \dfrac{n\pi}{c}$, $n = 1, 2, 3, \ldots$

For $\Delta F + (\xi - \mu_n^2)F = 0$ let $F(x,y) = S(x)T(y)$

$\rightarrow S''T + ST'' + (\xi - \mu_n^2)ST = 0 \quad \times \dfrac{1}{ST}$

$\rightarrow \dfrac{S''}{S} + (\xi - \mu_n^2) = -\dfrac{T''}{T} = \ell$

Now, $T'' + \ell T = 0$ with B.C. $T(0) = T(b) = 0$

so $\ell = \nu_j^2$ with $\nu_j = \dfrac{j\pi}{b}$, $j = 1, 2, 3, \ldots$

Finally, $S'' + (\xi - \mu_n^2 - \nu_j^2)S = 0$ with $S(0) = S(a) = 0$,

so $\xi - \mu_n^2 - \nu_j^2 = \lambda_k^2$ with $\lambda_k = \dfrac{k\pi}{a}$, $k = 1, 2, 3, \ldots$

$\therefore$ must have $\xi = \lambda_k^2 + \mu_n^2 + \nu_j^2 = \pi^2\left[\left(\dfrac{k}{a}\right)^2 + \left(\dfrac{n}{c}\right)^2 + \left(\dfrac{j}{b}\right)^2\right]$

and $E = \boxed{\overbrace{E_{k,n,j}}^{\text{e-values}} = \dfrac{\hbar^2}{2m}\xi = \dfrac{\hbar^2\pi^2}{2m}\left[\left(\dfrac{k}{a}\right)^2 + \left(\dfrac{n}{c}\right)^2 + \left(\dfrac{j}{b}\right)^2\right]}$

where $k, j, n = 1, 2, 3, \ldots$ i.e. energy is quantized.
(can only take on certain eigenvalues)

minimal value is $\boxed{E_{1,1,1} = \dfrac{\hbar^2\pi^2}{2m}\left[\dfrac{1}{a^2} + \dfrac{1}{c^2} + \dfrac{1}{b^2}\right]}$

#6 We'll do Problem 2 first. The sum of the kl and lk modes of a square membrane is given by

$$u_{kl}(x,y,t) + u_{lk}(x,y,t) = A_{kl}\sin\frac{k\pi x}{a}\sin\frac{l\pi y}{a}\cos(\omega_{kl}+\phi_{kl}) + A_{lk}\sin\frac{l\pi x}{a}\sin\frac{k\pi y}{a}\cos(\omega_{lk}+\phi_{lk})$$

Because $\omega_{kl} := \omega_{lk}$, the frequency of the sum is ω_{kl}. Now $c_1 u_{kl} + c_2 u_{lk}$ also has frequency ω_{kl}.