Assignment 6 **MATH 3C3** **21 November 2005**

1. The temperature in a circular plate of radius 1 is given by

$$\kappa\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t},\qquad 0<r<1,$$

with boundary condition $u(1,t)=0$, and initial condition $u(r,0)=f(r)$.

   (a) Use separation of variables to split the PDE into two ODEs.

   (b) Enforce the boundary condition to determine the allowed values of the separation constant. Write down the normal modes and the general solution.

   (c) Write down the integral expression for the constants in the series solution by imposing the initial condition. [Hint: $\int_0^1 [J_0(\alpha_m^{(0)}x)]^2 x\,\mathrm{d}x=\frac{1}{2}[J_1(\alpha_m^{(0)})]^2$ where $\alpha_m^{(0)}$ is the mth zero of the Bessel function J_0.]

2. The electric potential $u(x,y)$ in the semi-infinite strip $x>0$, $0<y<a$ satisfies the Laplace equation. Find the potential in the strip if $u(x,y)$ is finite throughout the strip and satisfies the insulator boundary conditions

$$u_y(x,0)=u_y(x,a)=0$$

on the top and bottom of the strip, and has the fixed potential

$$u(0,y)=\begin{cases}1, & 0\le y\le a/2,\\ 0, & a/2<y\le a\end{cases}$$

at $x=0$ on the $y-$axis at the end of the strip.

3. Consider Laplace's equation

$$\Delta u(x,y,z)=0\quad\text{for}\quad 0<x<L,\quad 0<y<M,\quad 0<z<N,$$

with boundary conditions

$$\frac{\partial u}{\partial x}(0,y,z)=\frac{\partial u}{\partial x}(L,y,z)=\frac{\partial u}{\partial y}(x,0,z)=\frac{\partial u}{\partial y}(x,M,z)=\frac{\partial u}{\partial z}(x,y,0)=0,$$

and

$$\frac{\partial u}{\partial z}(x,y,N)=f(x,y).$$

   (a) Use separation of variables to separate the partial differential equation into three ordinary differential equations (with their associated boundary conditions).

   (b) Find the normal modes by solving the ordinary differential equations you found in (a) using the first five boundary conditions. Write down the general solution.

   (c) Use the remaining boundary condition to write down the integral expression for the constants in the series solution.

(d) If $f(x,y) = \cos(5\pi x/L)\cos(3\pi y/M)$ find the series solution. [Hint: you do not need to evaluate any integrals!]

4. An infinite cylindrical rod of radius R has an initial temperature distribution depending *only* on the radial distance r. The temperature u satisfies the heat equation $c^2\nabla^2 u = \frac{\partial u}{\partial t}$ (in cylindrical coordinates $\nabla^2 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$).

   Suppose that the lateral surface of the rod is insulated, i.e. that $\frac{\partial u}{\partial r} = 0$ when $r = R$. Let u_∞ denote the steady state temperature of the rod, i.e. the (constant) temperature of the rod after a very long time has passed.

   Show that the temperature at time t is given by

$$u(r,t) = u_\infty + \sum_{m=1}^{\infty} A_m J_o(\beta_m r/R) \exp\left[-\left(\frac{\beta_m c}{R}\right)^2 t\right].$$

   Here the β_m denote the locations of the extrema of J_0, or in other words the points satisfying $J_0'(\beta_m) = 0$. Precise initial conditions have not been specified, so it is not possible to determine the values of the constants A_m without additional information. Hint: A physical argument is useful in determining the sign of the separation constant.

5. An atomic (quantum mechanical) particle is confined inside a rectangular box of sides a, b, c. The particle is described by a wave function ψ which satisfies the Schrödinger wave equation

$$-\frac{\hbar^2}{2m}\Delta\psi = E\psi.$$

   The wave function is required to vanish at each surface of the box (but not to be identically zero). This condition imposes constraints on the separation constants and therefore on the energy E. Show that the smallest value of E for which such a solution can be obtained is

$$E = \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

6. Text: §16.4 problems 2 and 3 (p. 785).