Math3C3 – A5

1. (a) $(1-x^2)y'' - xy' + n^2 y = 0$

$$\rightarrow y'' \frac{-x}{(1-x^2)} y' + \frac{n^2}{(1-x^2)} y = 0 \quad \circledast$$

let $p = \exp\left[\int P(x)dx\right] = \exp\left[-\int \frac{x}{1-x^2} dx\right]$

$= \exp\left[+\frac{1}{2}\log(1-x^2)\right] = (1-x^2)^{+1/2}$

multiply $\circledast$ by p →

$$\frac{d}{dx}\left[(1-x^2)^{1/2}\frac{dy}{dx}\right] + \frac{n^2}{(1-x^2)^{1/2}} y = 0$$

$\rightarrow W(x) = (1-x^2)^{-1/2}$, $P(x) = (1-x^2)^{1/2}$, $P(\pm 1) = 0$

B.C. $y_1^* \, p \, y_2' \Big|_{-1}^{1} - (y_1^*)' p \, y_2 \Big|_{-1}^{1} = 0$

since ~~p(x)~~ $p(\pm 1) = 0$ we require $y(\pm 1)$ & $y'(\pm 1)$ to be finite.

(b) V_1 does not obey BC since $V_1'(\pm 1)$ is not finite ∴ B.C. criterion for s.a. S-L problem is not satisfied.

2.(a) $(1-x^2)U_n'' - 3xU_n' + n(n+2)U_n = 0$

put into std. form :

$$(*) \quad U_n'' - \frac{3x}{(1-x^2)}U_n' + \frac{n(n+2)}{(1-x^2)}U_n = 0$$

Regular singular pts at $x = \pm 1 \rightarrow$ B.C $\boxed{|U_n(\pm 1)| < \infty.}$

transform to S-L form by multiplying by

$$p(x) = \exp\Big[\underbrace{\int \frac{-3x}{(1-x^2)}dx}_{\substack{\text{let } \mu = 1-x^2 \\ d\mu = -2x\,dx}}\Big] = \exp\Big[\tfrac{3}{2}\int \frac{d\mu}{\mu}\Big] = \exp\Big[\tfrac{3}{2}\ln\mu\Big]$$

$$\Rightarrow \boxed{p(x) = (1-x^2)^{3/2}}$$

$$(*) \rightarrow \boxed{\frac{d}{dx}\Big[\underbrace{(1-x^2)^{3/2}}_{p(x)}\frac{dU_n}{dx}\Big] + \underbrace{n(n+2)}_{\text{e-values}}\underbrace{(1-x^2)^{1/2}}_{w(x)}U_n = 0}$$

(b). from part (c) $w(x) = (1-x^2)^{1/2}$, $a = -1$, $b = +1$

$\therefore$ e-fcns U_n, U_m satisfy the following orthogonality condition since L is s.a. with B.C. $|U_n(\pm 1)| < \infty$ ~~($+|U_n'(\pm 1)| < \infty$)~~

$$\langle U_n, U_m\rangle_w = \int_{-1}^{1} U_n(x)U_m(x)(1-x^2)^{1/2}dx = \begin{cases} 0 & \text{if } m \neq n \\ \|U_n(x)\|_w^2 & \text{if } m = n. \end{cases}$$

3. Write $v = u'$ & differentiate equation:

$\rightarrow \frac{d}{dx}\left[p'v + pv'\right] + \lambda v = 0$

$\rightarrow pv'' + 2p'v' + p''v + \lambda v = 0$

std. form: $v'' + \frac{2p'}{p}v' + \frac{p''}{p}v + \frac{\lambda}{p}v = 0$

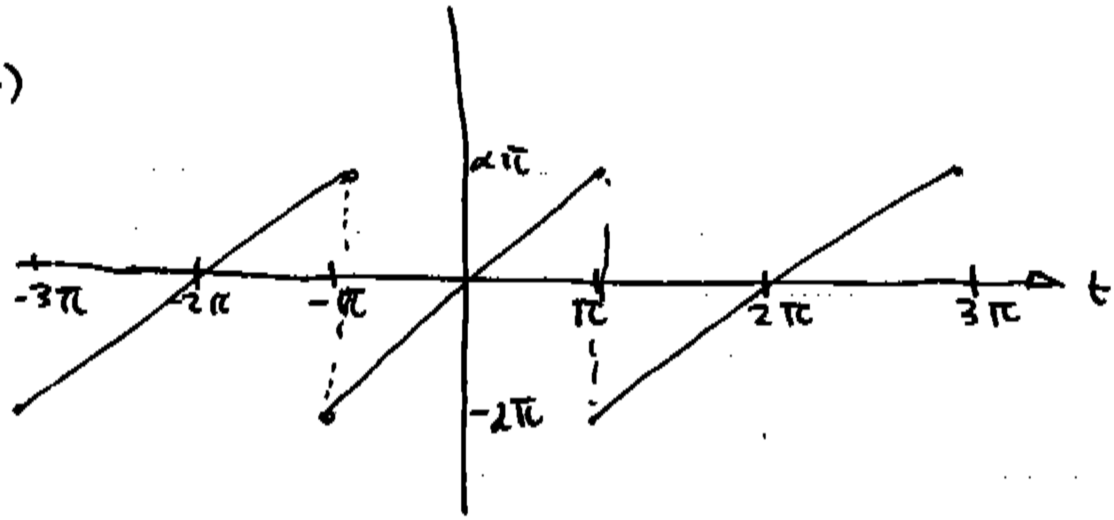
multiply by $P(x) = \exp\left[\int \frac{2p'}{p}dx\right] = p^2$

$\rightarrow \frac{d}{dx}\left(p^2 \frac{dv}{dx}\right) + p''pv + \lambda pv = 0$ (S-L form)

$\therefore$ s.a. with suitable B.C.
& weight function $W(x) = p(x)$

$\therefore \langle v_m, v_n \rangle_p = \langle u_m', u_n' \rangle_p = 0 \quad (m \neq n)$

4. (a)



(b) Use Normalized Fourier series on [-π, π]

$$y_n(t) = \sqrt{\frac{1}{\pi}} \sin nt \qquad \text{(only sine since } \alpha(t) \text{ is an \underline{odd} function)}$$

$$c_n = \langle y_n, \alpha \rangle = \int_{-\pi}^{\pi} \frac{\alpha t}{\sqrt{\pi}} \sin nt \, dt$$

use int. by parts with $u = \frac{\alpha t}{\sqrt{\pi}}$ $\quad dV = \sin nt\,dt$

$du = \frac{\alpha dt}{\sqrt{\pi}} \quad V = \frac{1}{n}\cos nt$

$$c_n = -\frac{\alpha t}{\sqrt{\pi}\, n} \cos nt \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\alpha}{\sqrt{\pi}\, n} \cos nt \, dt$$

$$= -\frac{\alpha t}{\sqrt{\pi}\, n} \cos nt \Big|_{-\pi}^{\pi} + \frac{\alpha}{\sqrt{\pi}\, n^2} \cancel{\sin nt}^{\,0} \Big|_{-\pi}^{\pi}$$

$$= \frac{2\alpha\sqrt{\pi}}{n} (-1)^{n+1}$$

$$\therefore \quad \alpha t = 2\alpha\sqrt{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{\pi}} \sin nt$$

or $\boxed{t = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt}$

4 . (c) subs. $t = \frac{\pi}{2} \Rightarrow \frac{\pi}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k}}{(2k+1)}$ (series terms $= 0$ for n even, $= (-1)^k$ for $n = 2k+1$ n odd)

$\Rightarrow \boxed{\pi = 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)}}$

(d) Parseval: $\|Q\|^2 = \sum_{n=1}^{\infty} |c_n|^2$

$\|Q\|^2 = 4\sum_{n=1}^{\infty} \frac{1}{n^2}\pi$

$\Rightarrow \int_{-\pi}^{\pi} t^2\,dt = \frac{1}{3}\left(\pi^3 - (-\pi)^3\right) = \frac{2}{3}\pi^3 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\Rightarrow \pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6} \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$

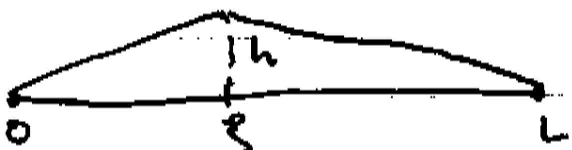
5. 1D Wave eq. : $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

s.o.v. : $u(x,t) = S(x)\,G(t)$

$\rightarrow\ S''G = \frac{1}{c^2} S G''$

$\rightarrow\ \frac{S''}{S} = \frac{1}{c^2}\frac{G''}{G} = \ell$ (const.)

$\rightarrow$ ① $S'' = \ell S$
② $G'' = \ell c^2 G$



I.C. , B.C. $u(0,t)=0$
$u(L,t)=0$

to ① to fit in with B.C. $\ell = -\nu^2$

$\rightarrow\ S(x) = A\sin\nu x + B\cos\nu x$

$S(0)=0 \rightarrow B=0$, $S(L)=0 \rightarrow \nu_n L = n\pi$, $n=0,1,2$

$\rightarrow\ S_n(x) = \sin\frac{n\pi x}{L}$

② $\rightarrow\ G'' = -(\nu_n c)^2 G$

$\rightarrow\ G(t) = A'\sin\nu_n ct + B'\cos\nu_n ct$

$G'(t) = 0$ ("motionless")

$\rightarrow\ \boxed{A'=0}$

$\therefore$ normal modes are

$u_n(x,t) = A_n \sin\frac{n\pi x}{L}\cos\left(\frac{n\pi c t}{L}\right)$

& general solution is $u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\left(\frac{n\pi c t}{L}\right)$

5. (cont.)

impose I.C. to find A_n: take inner product of both sides w.r.t. $\sin\frac{m\pi x}{L}$

$$\Rightarrow \int_0^{\xi} \frac{h}{\xi}\, x \sin\frac{m\pi x}{L}\,dx + \int_{\xi}^{L} \frac{h}{(L-\xi)}(L-x)\sin\frac{m\pi x}{L}\,dx$$

$$= A_m \frac{L}{2}$$

$$= \frac{2h}{\xi}\,\frac{L}{m^2\pi^2}\left[L\sin\frac{m\pi\xi}{L} - \cancel{m\pi\xi\cos\left(\frac{m\pi\xi}{L}\right)}\right]$$

$$+ \frac{2h}{L-\xi}\,\frac{L}{m^2\pi^2}\left[\cancel{(L-\xi)m\pi\cos\left(\frac{m\pi\xi}{L}\right)} + L\sin\frac{m\pi\xi}{L}\right]$$

$$\Rightarrow A_m = \frac{2hL}{m^2\pi^2}\sin\frac{m\pi\xi}{L}\left[\frac{L-\xi+\xi}{\xi(L-\xi)}\right]$$

$$= \frac{2hL^2}{m^2\pi^2\,\xi(L-\xi)}\sin\frac{m\pi\xi}{L}$$

$$\therefore \boxed{u(x,t) = \frac{2hL^2}{\pi^2\,\xi(L-\xi)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi\xi}{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}}$$

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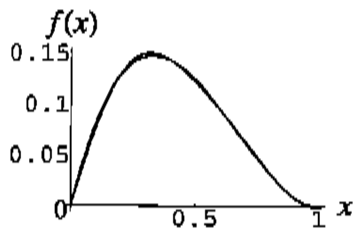
6. The normalized eigenfunctions are $\phi_n(x) = 2^{1/2} \sin \frac{(2n-1)\pi x}{2}$ for $n = 1, 2, \ldots$.

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) = 2^{1/2} \sum_{n=1}^{\infty} a_n \sin \frac{(2n-1)\pi x}{2}$$

$$\begin{aligned} a_n &= 2^{1/2} \int_0^1 x(1-x)^2 \sin \frac{(2n-1)\pi x}{2}\, dx = 2^{1/2} \left\{ \frac{32}{\pi^3} \left[\frac{1}{(2n-1)^3} - \frac{3/\pi}{(2n-1)^4} (-1)^{n+1} \right] \right\} \\ &= 2^{1/2} \frac{32}{\pi^3} \frac{1}{(2n-1)^3} \left[1 + \frac{(-1)^n (3/\pi)}{2n-1} \right] \end{aligned}$$

$$f(x) = \frac{64}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \left[1 + \frac{(-1)^n (3/\pi)}{2n-1} \right] \sin \frac{(2n-1)\pi x}{2}$$



The accompanying figure shows $f(x)$ plotted against x using four and six terms along with $x(1-x)^2$.