**Assignment 5** **MATH 3C** **7 November 2005**

1. Consider the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

   (a) Put this equation into Sturm–Liouville form, and indicate which weight function and boundary conditions on $[-1,1]$ make it self-adjoint. The boundary conditions need special attention.

   (b) The functions $T_0(x) = 1$ and $V_1(x) = (1-x^2)^{1/2}$ are both solutions to the differential equation, corresponding to different eigenvalues. Explain why they are nevertheless not orthogonal with respect to the relevant weight function.

2. (a) The Chebyshev polynomials satisfy the differential equation

$$(1-x^2)\, U_n''(x) - 3x\, U_n'(x) + n(n+2)\, U_n(x) = 0.$$

   Put this equation into Sturm-Liouville form and identify the weight function, eigenvalues and appropriate boundary conditions.

   (b) State the possible values of the following integral

$$\int_{-1}^{1} U_m(x)\, U_n(x)\, (1-x^2)^{1/2}\, \mathrm{d}x.$$

   You must justify your answer.

3. The functions $u_n(x)$ satisfy the Sturm–Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}u_n(x)\right] + \lambda_n u_n(x) = 0$$

   and obey boundary conditions that make this equation self-adjoint and hence lead to orthogonal eigenfunctions. For disjoint eigenvalues $\lambda_m \neq \lambda_n$, show that under appropriate boundary conditions $u_m'(x)$ and $u_n'(x)$ are orthogonal with weight function $p(x)$.

4. Consider the function

$$Q(t) = \alpha t, \qquad -\pi \leq t \leq \pi$$

   (a) Sketch the waveform on the interval $[-3\pi, 3\pi]$ (assume $Q(t)$ is periodic).

   (b) Find the Fourier coefficients of $Q(t)$ (remember to normalize the Fourier basis functions correctly). Why are the coefficients of the cos series equal to zero?

   (c) Use the series you found in (b) to show that

$$\pi = 4\sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}.$$

   [Hint: evaluate $Q(t)$ at $t = \pi/2$.]

(d) Use your result from (b) with Parseval's theorem to show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

5. A stretched string of length $L$ is plucked at the point $\xi$ $(0 < \xi < L)$ so that its initial shape is given by

$$u(x,0) = \begin{cases} \frac{hx}{\xi} & 0 \le x \le \xi \\ \frac{h(L-x)}{L-\xi} & \xi \le x \le L \end{cases}$$

The string is held motionless in this configuration and then released. Solve the one-dimensional wave equation using separation of variables, to obtain

$$u(x,t) = \frac{2hL^2}{\pi^2 \xi (L-\xi)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi\xi}{L} \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}.$$

6. Text: §14.4 problem 3 (p. 701).