Math3C A4 Solutions ①

1. ① $\psi \propto e^{-\frac{x^2}{2\sigma^2}}$

useful integrals: $\langle \psi, x\psi \rangle = \int_{-\infty}^{\infty} x|\psi|^2 dx = 0$ ($x|\psi|^2$ odd.)

$\langle \psi, p\psi \rangle = \frac{\hbar}{i} \int_{-\infty}^{\infty} \psi(x) \frac{\hbar}{i}\frac{d\psi}{dx} dx = 0$ (ψ even, $\frac{d\psi}{dx}$ odd $\to \psi\frac{d\psi}{dx}$ odd)

∴ only need to compute $\langle \psi, x^2\psi \rangle$ & $\langle \psi, p^2\psi \rangle$

$$\langle \psi, x^2\psi \rangle = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{\pi}} e^{-x^2/\sigma^2} dx = \frac{\sigma^2}{2}$$

$$p^2\psi = -\hbar^2 \psi''(x) = \hbar^2 \frac{d}{dx}\left(\frac{1}{\sigma^{1/2}\pi^{1/4}}\left(\frac{-x}{\sigma}\right) e^{-x^2/2\sigma^2}\right)$$

$$= \frac{\hbar^2}{\pi^{1/4}\sigma^{5/2}} e^{-x^2/2\sigma^2}\left(1-\frac{x^2}{\sigma^2}\right)$$

$$\therefore \langle \psi, p^2\psi \rangle = \frac{\hbar^2}{\sqrt{\pi}\,\sigma^3}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\left(1-\frac{x^2}{\sigma^2}\right)dx$$

$$= \frac{\hbar^2}{2\sigma^2}$$

∴ Finally, $\langle \psi, (x-\cancel{\langle \psi, x\psi\rangle})^2\psi\rangle\langle \psi, (p-\cancel{\langle \psi, p\psi\rangle})^2\psi\rangle$

$$= \langle \psi, x^2\psi\rangle\langle \psi, p^2\psi\rangle$$

$$= \frac{\sigma^2}{2}\frac{\hbar^2}{2\sigma^2} = \frac{\hbar^2}{4}$$

∴ Gaussian wavepacket gives the minimum possible product of uncertainties. This also gives a concrete interpretation of uncertainty ~~as~~ → width) σ of wavepacket.

$$x y'' + (1+x)y' + y = 0$$

2. (a) std form: $y'' + \left(\frac{1+x}{x}\right) y' + \frac{1}{x} y = 0$

→ singular pt. at $x=0$ only

$\lim_{x\to 0} x \cdot \frac{(1+x)}{x} = 1 < \infty$ & $\lim_{x\to\infty} x^2 \frac{(1+x)}{x} = 0 < \infty$

→ $\boxed{\text{Regular singular pt. at } x=0}$

(b) $\boxed{2}$ let $y(x) = \sum_{k=0}^{\infty} a_k x^{k+s}$ sub in ODE

$\to \sum_{k=0}^{\infty} a_k (k+s)(k+s-1) x^{k+s-1} + (1+x) \sum_{k=0}^{\infty} a_k (k+s) x^{k+s-1}$

$+ \sum_{k=0}^{\infty} a_k x^{k+s} = 0$

lowest power: x^{s-1}: $a_0 s(s-1) + a_0 s = 0$ $(a_0 \neq 0)$

$\to s^2 = 0 \to \boxed{s=0}$

Equal root → Frobenius will give only one solution

(c) sub in $s=0$:

$\to \sum_{k=1}^{\infty} a_k k(\cancel{k-1}) x^{k-1} + (1+\cancel{x}) \sum_{k=0}^{\infty} a_k k x^{k-1}$

$+ \sum_{k=0}^{\infty} a_k x^k = 0$

$\underbrace{\sum_{k=1}^{\infty} a_k k^2 x^{k-1}}_{\substack{\text{shift indices} \\ k' = k-1}} + \sum_{k=0}^{\infty} a_k (k+1) x^{k} = 0$

$\sum_{k=0}^{\infty} a_{k+1} (k+1)^2 x^k$

2 (cont.)

Recurrence relation: $\boxed{a_{k+1} = -\frac{1}{(k+1)} a_k}$

$\rightarrow a_1 = \frac{-1}{1} a_0, \quad a_2 = \frac{(-1)}{2} a_1 = \frac{(-1)^2}{1\cdot 2} a_0, \quad a_3 = \frac{(-1)^3}{1\cdot 2\cdot 3} a_0$

$\boxed{a_k = \frac{(-1)^k}{k!}} \quad \therefore \quad \boxed{y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k}$

or $\boxed{y_1(x) = e^{-x}}$

(d) Since we have an equal root, use formula

(Note: can also use integral formula $y_2 = y_1 \int \frac{\exp(-\int P(x)dx)}{y_1^2} dx = e^{-x} \int \frac{e^x}{x} dx$ expand ... but this is more harder)

$y_2(x) = y_1(x) \log x + \sum_{j=0}^{\infty} c_j x^{j+r}$

$\boxed{y_2(x) = e^{-x} \log x + \underbrace{\sum_{j=0}^{\infty} c_j x^{j+r}}_{v(x)}}$ sub in eq:

$x\left[\cancel{e^{-x}\log x} - 2\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2}\right] + (1+x)\left[-\cancel{e^{-x}\log x} + \frac{e^{-x}}{x} + v'\right]$
$+ v''$
$+ \cancel{e^{-x}\log x} + v = 0$

$\rightarrow \boxed{x v'' + (1+x) v' + v = e^{-x}}$

$\rightarrow \sum_{j=0}^{\infty} c_j (j+r)(j+r-1) x^{j+r-1} + (\cancel{1}+x) \sum_{j=0}^{\infty} c_j (j+r) x^{j+r-1} + \sum_{j=0}^{\infty} c_j x^{j+r}$

$= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^j$

$\rightarrow \boxed{\sum_{j=0}^{\infty} c_j (j+r)^2 x^{j+r-1} + \sum_{j=0}^{\infty} c_j (j+r+1) x^{j+r} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^j}$

2 (cont.

lowest power determines r:

L.H.S. x^{r-1} R.H.S. x^0 $\rightarrow r-1=0 \rightarrow \boxed{r=1}$ sub in:

$$\sum_{j=0}^{\infty} c_j (j+1)^2 x^j + \underbrace{\sum_{j=0}^{\infty} c_j (j+2) x^{j+1}}_{\text{shift:}} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^j$$

$$\sum_{j=1}^{\infty} c_{j-1} (j+1) x^j$$

x^0: $\boxed{c_0 = 1}$

$x^j,\ j \geq 1$: $c_j (j+1)^2 + c_{j-1}(j+1) = \dfrac{(-1)^j}{j!}$

$$\rightarrow \boxed{c_j = -\frac{c_{j-1}}{(j+1)} + \frac{(-1)^j}{j!(j+1)^2}}$$

$$\rightarrow c_1 = -\frac{c_0}{2} + \frac{(-1)}{4} = -\frac{1}{2} - \frac{1}{4} = \boxed{-3/4 = c_1}$$

General solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$\boxed{y(x) = c_1 e^{-x} + c_2 \left[e^{-x} \log x + x - 3/4\, x^2 + \cdots \right]}$$

3. (a) Put equation into std. form $y'' + P(x)y' + Q(x)y$

$$\rightarrow y'' + \underbrace{\frac{(x-2)}{x}}_{P(x)} y' + \underbrace{\frac{(x^2+2)}{x^2}}_{Q(x)} y = 0$$

$\lim_{x\to 0} P(x) = \lim_{x\to 0} Q(x) = \infty \quad \therefore x=0$ is a s.p.

$\lim_{x\to 0} xP(x) = -2 < \infty$ & $\lim_{x\to 0} x^2 Q(x) = 2 < \infty$

$\therefore x=0$ is a r.s.p.

(b) let $y(x) = \sum_{k=0}^{\infty} a_k x^{k+s}$ (Frobenius method)

sub in equation:

$$\sum_{k=0}^{\infty} a_k (k+s)(k+s-1) x^{k+s} + \sum_{k=0}^{\infty} a_k (k+s) x^{k+s+1}$$
$$- \sum_{k=0}^{\infty} 2 a_k (k+s) x^{k+s} + \sum_{k=0}^{\infty} a_k x^{k+s+2}$$
$$+ \sum_{k=0}^{\infty} 2 a_k x^{k+s} = 0$$

$$\rightarrow \sum_{k=0}^{\infty} a_k [(k+s)(k+s-3)+2] x^{k+s} + \sum_{k=0}^{\infty} a_k (k+s) x^{k+s+1}$$
$$+ \sum_{k=0}^{\infty} a_k x^{k+s+2} = 0$$

Lowest power: x^s ($k=0$ in first term)

$\rightarrow a_0[s(s-3)+2] = 0$ (assume $a_0 \neq 0$)

$\rightarrow s^2 - 3s + 2 = 0 \rightarrow \boxed{(s-2)(s-1) = 0 \text{ "indicial eqn"}}$

Roots are $\boxed{s=2,\ s=1}$, since roots differ by an integer larger root will give a soln, smaller root may or may not.

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(c) sub in $s=2$ (larger root):

$$\sum_{k=0}^{\infty} a_k\left[(k+2)(k-1)+2\right]x^{k+2} + \sum_{k=0}^{\infty} a_k(k+2)\,x^{k+3}$$
$$+\sum_{k=0}^{\infty} a_k x^{k+4} = 0$$

x^2 lowest power : a_0 (arbitrary)

shift indices:

$$\cancel{\sum_{k=0}^{\infty} a_k\left[(k+2)(k-1)+2\right.}\quad \sum_{k=0}^{\infty} a_k k(k+1)\,x^{k+2}$$
$$+\sum_{k=1}^{\infty} a_{k-1}(k+1)\,x^{k+2} + \sum_{k=2}^{\infty} a_{k-2}\,x^{k+2} = 0$$

Recurrence relation:

$$a_k = -\frac{a_{k-1}}{k} - \frac{a_{k-2}}{k(k+1)}, \quad k \geq 2$$

x^3: $a_1 2 + a_0 2 = 0 \Rightarrow \boxed{a_1 = -a_0}$

x^4: $a_2 = -\frac{a_1}{2} - \frac{a_0}{2\cdot 3} = a_0\left[+\frac{1}{2} - \frac{1}{6}\right] = +\frac{1}{3}$

$\therefore \boxed{y_1(x) = a_0\left[x^2 - x^3 + \frac{1}{3}x^4 + \cdots\right]}$

(d) second solution: does $s=1$ yield a Frobenius solution? sub in $s=1$ in series:

$$\rightarrow \sum_{k=0}^{\infty} a_k \left[(k+1)(k-2)+2\right] x^{k+1} + \sum_{k=0}^{\infty} a_k (k+1) x^{k+2}$$

$$+ \sum_{k=0}^{\infty} a_k x^{k+3} = 0$$

shift indices:

$$\sum_{k=0}^{\infty} a_k\, k(k-1) x^{k+1} + \sum_{k=1}^{\infty} a_{k-1}\, k\, x^{k+1}$$

$$+ \sum_{k=2}^{\infty} a_{k-2}\, x^{k+1} = 0$$

∴ recurrence relation is

$$a_k = \underbrace{-\frac{a_{k-1}}{k-1}}_{k \geq 1} \overbrace{- \frac{a_{k-2}}{k(k-1)}}^{k \geq 2}$$

↳ blows up at $k=1$!

∴ Frobenius only gives <u>one</u> solution (for $s=2$)

→ Look for solution of form

$$\boxed{y_2 = c\, y_1 \log x + \sum_{j=0}^{\infty} c_j\, x^{j+1}}$$

$(c \neq 0\,!)$

4. (a) The complementary solution is $y_c(x) = c_1 \cos x + c_2 \sin x$. The particular solution is of the form $y_p(x) = \alpha e^x$. Substituting $y_p(x) = \alpha e^x$ into the differential equation gives $(\alpha + \alpha)e^x = e^x$ or $\alpha = 1/2$. Thus, $y_p(x) = \frac{1}{2}e^x$.

(b) The complementary solution is $y_c(x) = c_1 e^x + c_2 e^{2x}$. The particular solution is of the form $y_p(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$. Substituting $y_p(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$ into the differential equation gives

$$2\gamma + 6\delta x - 3\beta - 6\gamma x - 9\delta x^2 + 2\alpha + 2\beta x + 2\gamma x^2 + 2\delta x^3 = x^3$$

from which we find that $2\delta = 1$, $2\gamma - 9\delta = 0$ ($\gamma = 9/4$), $6\delta - 6\gamma + 2\beta = 0$ ($\beta = 21/4$), and $2\gamma - 3\beta - 2\alpha = 0$ ($\alpha = 45/8$). Thus, $y_p(x) = \frac{45}{8} + \frac{21}{4}x + \frac{9}{4}x^2 + \frac{1}{2}x^3$.

5. Put $r = -1/2$ in the term in braces in Equation 4 to get $2n(n - \frac{1}{2})c_n = c_{n-1}$ or $c_n = -c_{n-1}/n(2n-1)$ for $n \geq 1$. This formula gives us

$$c_1 = \frac{c_0}{1} = c_0, \quad c_2 = \frac{c_1}{2\cdot 3} = \frac{c_0}{3!} = \frac{2^2 c_0}{4!}, \quad c_3 = \frac{c_2}{3\cdot 5} = \frac{2^2 c_0}{3\cdot 5!} = \frac{2^3 c_0}{6!}, \quad \text{and so on}$$

6. From Equation 20, we find that

$$\begin{aligned}
y_1(x) &= x^{1/2}\left[1 - \frac{3}{4}x + \frac{9}{64}x^2 - \frac{3}{256}x^3 + O(x^4)\right] \\
y_1^2(x) &= x\left[1 - \frac{3}{2}x + \left(\frac{9}{16} + \frac{9}{32}\right)x^2 - \left(\frac{6}{256} + \frac{27}{128}\right)x^3 + O(x^4)\right] \\
&= x\left[1 - \frac{3}{2}x + \frac{27}{32}x^2 - \frac{15}{64}x^3 + O(x^4)\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{y_1^2(x)} &= \frac{1}{x}\left[1 + \frac{3}{2}x - \frac{27}{32}x^2 + \frac{15}{64}x^3 + \cdots + \frac{9}{4}x^2 - \frac{81}{32} + \cdots + \frac{27}{8}x^3 + O(x^4)\right] \\
&= \frac{1}{x}\left[1 + \frac{3}{2}x + \frac{45}{32}x^2 + \frac{69}{64}x^3 + O(x^4)\right]
\end{aligned}$$

Now $\int p\,dx = 0$, so

$$\begin{aligned}
y_2(x) &= y_1(x)\int \frac{dx}{y_1^2(x)} = y_1(x)\left[\ln x + \frac{3}{2}x + \frac{45}{64}x^2 + \frac{23}{64}x^3 + O(x^4)\right] \\
&= y_1(x)\ln x + x^{3/2}\left[\frac{3}{2} + \left(\frac{45}{64} - \frac{9}{8}\right)x + \left(\frac{23}{64} - \frac{135}{256} + \frac{27}{128}\right)x^2 + O(x^3)\right] \\
&= y_1(x)\ln x + x^{3/2}\left[\frac{3}{2} - \frac{27}{64}x + \frac{11}{256}x^2 + O(x^3)\right]
\end{aligned}$$