Assignment 4 **MATH 3C** **24 October 2005**

1. Let X denote the position operator (multiplication by x) and let $P = \frac{\hbar}{i}\frac{d}{dx}$ denote the momentum operator. The following uncertainty principle was derived in class: for any normalized wave function ψ,

$$\langle \psi, (X - \langle \psi, X\psi\rangle)^2 \psi\rangle \langle \psi, (P - \langle \psi, P\psi\rangle)^2 \psi\rangle \geq \frac{\hbar^2}{4}.$$

Show that the above inequality is actually an equality when ψ is the normalized Gaussian wave function $\psi(x) = \pi^{-1/4}\sigma^{-1/2}e^{-x^2/2\sigma^2}$, for any $\sigma > 0$. (Hint: $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$, $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}/2$.)

2. Consider the ODE

$$x\,y''(x) + (1+x)\,y'(x) + y(x) = 0.$$

   (a) Find and classify the singular points of the ODE.

   (b) Solve the indicial equation associated to the Frobenius solution of the ODE about the point $x_0 = 0$. How many linearly independent solutions do you expect the Frobenius method to give?

   (c) Solve the recurrence relation arising from the larger root of the indicial equation. Express the result in terms of an elementary function.

   (d) Find the recurrence relation and the first two terms in the series solution for the second linearly independent solution of the ODE. Write down the general solution for the ODE.

3. Consider the second-order homogeneous linear differential equation

$$x^2 y'' + x(x-2)y' + (x^2+2)y = 0.$$

   (a) Show that $x = 0$ is a regular singular point.

   (b) Find the roots of the indicial equation for the series solution about $x = 0$. How many linearly independent solutions do you expect the method of Frobenius to yield? Why?

   (c) Calculate the first three terms of one solution using the method of Frobenius.

   (d) Give the precise form of the second linearly independent solution. Justify your answer. Do not calculate the coefficients.

4. Text: §11.4 problem 6 (p. 554).

5. Text: §12.4 problem 3 (p. 599).

6. Text: §12.4 problem 15 (p. 599).