# A3 - Math 3C

1. (a) $A$ is s.a. $\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(1-\lambda)-1\right] + (\lambda-1) = 0$$

$$= (\lambda-1)\left[1-(2-\lambda)(1-\lambda) + 1\right] = 0$$

$$\Rightarrow (\lambda-1)\left[2 - 2 + 3\lambda - \lambda^2\right] = (\lambda-1)\lambda(\lambda-3) = 0$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = +3 \qquad \text{e-vals.}$$

e-vec. $\xi_1$: $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \rightarrow \xi_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\xi_2$: $\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \rightarrow \xi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\xi_3$: $\left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \rightarrow \xi_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$\therefore U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

1.(b) $A$ is s.a. $\begin{vmatrix} 3-\lambda & 2i \\ -2i & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)^2 - 4 = 0 \Rightarrow 9 - 6\lambda + \lambda^2 - 4 = 0$
$\Rightarrow \lambda^2 - 6\lambda + 5 = 0 \quad \Rightarrow (\lambda-5)(\lambda-1) = 0 \quad \boxed{\lambda_1 = 5,\ \lambda_2 = 1}$

$\xi_1 : \left(\begin{array}{cc|c} -2 & 2i & 0 \\ -2i & -2 & 0 \end{array}\right) \Rightarrow \xi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

$\xi_2 : \left(\begin{array}{cc|c} 2 & 2i & 0 \\ -2i & 2 & 0 \end{array}\right) \Rightarrow \xi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\therefore\ U = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix},\ D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}.$

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2. (a) Let $\vec{x}(t) = \vec{a}\sin(\omega t + \delta)$ (normal modes)

$\Rightarrow \ddot{\vec{x}} = -\omega^2 \vec{a}\,\cancel{\sin(\omega t + \delta)} = A\vec{a}\,\cancel{\sin(\omega t + \delta)}$

$\therefore$ e-value problem is $\boxed{A\vec{a} = -\omega^2\vec{a}}$
ie. e-values are $-\omega^2$
e-vectors are $\vec{a}$

(b) since <u>A is symmetric</u> (ie. self adjoint) it has <u>n l.i. e-vectors</u> & hence there are <u>n normal modes</u>.

2 (a) let $\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, $\vec{y}_0 = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$

then can write system as

$$\dot{\vec{y}}(t) = A\vec{y}(t) \quad \text{with I.C. } \vec{y}(0) = \vec{y}_0$$

(b) $e^{At} = U e^{Dt} U^{+}$ since $A$ is s.a.

Find e-vals: $\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0 \;\to\; -(3-\lambda)(3+\lambda) - 16 = 0$

$\to -9 + \lambda^2 - 16 = 0$

$\to \lambda^2 = 25 \to \boxed{\lambda_{1,2} = \pm 5}$

$\xi_1$: $\left(\begin{array}{cc|c} -2 & 4 & 0 \\ 4 & -8 & 0 \end{array}\right) \to \xi_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\xi_2$: $\left(\begin{array}{cc|c} 8 & 4 & 0 \\ 4 & 2 & 0 \end{array}\right) \to \xi_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\therefore e^{At} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-5t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2e^{5t} & e^{5t} \\ -e^{-5t} & 2e^{-5t} \end{pmatrix}$$

$$\boxed{e^{At} = \frac{1}{5}\begin{pmatrix} 4e^{5t} + e^{-5t} & 2e^{5t} - 2e^{-5t} \\ 2e^{5t} - 2e^{-5t} & e^{5t} + 4e^{-5t} \end{pmatrix}}$$

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3. By def $e^{At} = I + At + \frac{(At)^2}{2} + \cdots$

$$A^2 = \begin{bmatrix} 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for } n \geq 4.$$

$$\therefore e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} \quad \underline{\text{exactly}}!$$

$$= \begin{bmatrix} 1 & t & 2t + \frac{3t^2}{2} & 2t + \frac{3t^2}{2} + \frac{1}{2}t^3 \\ 0 & 1 & 3t & t + 3t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & 2t + 3/2\, t^2 & 2t + 3/2\, t^2 + 1/2\, t^3 \\ 0 & 1 & 3t & t + 3/2\, t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4(a) $BC = CB$

$$Be^{Ct} = B\left[\sum_{j=0}^{\infty} \frac{(Ct)^j}{j!}\right] = \sum_{j=0}^{\infty} \frac{BC^j t^j}{j!}$$

but $BC^j = CBC^{j-1} = C^2 B C^{j-2} = \cdots = C^j B$

since $BC = CB$

$$\therefore \quad Be^{Ct} = \sum_{j=0}^{\infty} \frac{C^j t^j}{j!} B = e^{Ct} B \checkmark$$

(b)

$$e^{CBC^{-1}} = \sum_{j=0}^{\infty} \frac{(CBC^{-1})^j}{j!} = \sum_{j=0}^{\infty} \frac{\overbrace{(CBC^{-1})(CBC^{-1})\cdots(CBC^{-1})}^{j \text{ times}}}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{CB^jC^{-1}}{j!} = C\left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right) C^{-1}$$

(since C's don't depend on j)

$$= C e^{B} C^{-1} \checkmark$$

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5. (a)

$$I_{xx} = \sum_{i=1}^{3} m_i(y_i^2 + z_i^2) = 20$$

$$I_{yy} = \sum_{i=1}^{3} m_i(x_i^2 + z_i^2) = 18$$

$$I_{zz} = \sum_{i=1}^{3} m_i(x_i^2 + y_i^2) = 18$$

$$I_{xy} = I_{yx} = -\sum_{i=1}^{3} m_i x_i y_i = 0, \quad I_{xz} = I_{zx} = -\sum_{i=1}^{3} m_i x_i z_i = 0$$

$$I_{yz} = I_{zy} = -\sum_{i=1}^{3} m_i y_i z_i = -10$$

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$$I = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 18 & -10 \\ 0 & -10 & 18 \end{bmatrix}$$ inertia matrix.

6(b) $$\begin{vmatrix} 20-\lambda & 0 & 0 \\ 0 & 18-\lambda & -10 \\ 0 & -10 & 18-\lambda \end{vmatrix} = 0 = (20-\lambda)\left[(18-\lambda)^2-100\right] = 0$$
$$= (20-\lambda)(\lambda-28)(\lambda-8)$$

$\rightarrow \boxed{\lambda_1 = 20,\ \lambda_2 = 28,\ \lambda_3 = 8}$

$\lambda_1 = 20 : \vec{\xi}_1 : \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & -10 & 0 \\ 0 & -10 & -2 & 0 \end{array}\right] \rightarrow \vec{\xi}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 28 : \vec{\xi}_2 : \left[\begin{array}{ccc|c} -8 & 0 & 0 & 0 \\ 0 & -10 & -10 & 0 \\ 0 & -10 & -10 & 0 \end{array}\right] \rightarrow \vec{\xi}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$\lambda_3 = 8 : \vec{\xi}_3 : \left[\begin{array}{ccc|c} 12 & 0 & 0 & 0 \\ 0 & 10 & -10 & 0 \\ 0 & -10 & 10 & 0 \end{array}\right] \rightarrow \vec{\xi}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

**6.** (a) The eigenvalues of $S = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$ are $\lambda_1 = 3 - 2\sqrt{2}$ and $\lambda_2 = 3 + 2\sqrt{2}$. Both eigenvalues are positive, so the quadratic form is positive definite.

(b) The eigenvalues of $S = \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix}$ are $\lambda_1 = 2 - \sqrt{10}$ and $\lambda_2 = 2 + \sqrt{10}$, and so the quadratic form is not positive definite.

(c) The eigenvalues of $S = \begin{pmatrix} 2 & -6 \\ -6 & 5 \end{pmatrix}$ are $\lambda_1 = \dfrac{7 - 3\sqrt{17}}{3}$ and $\lambda_2 = \dfrac{7 + 3\sqrt{17}}{3}$, and so the quadratic form is not positive definite.