Assignment 3 **MATH 3C3** **10 October 2005**

1. Find a unitary diagonalization of matrix A, i.e. a unitary matrix U and a diagonal matrix D such that $D = U^\dagger A U$, for the following matrices

   (a)
$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

   (b)
$$A = \begin{bmatrix} 3 & 2i \\ -2i & 3 \end{bmatrix}.$$

2. Consider the following system of ordinary differential equations,
$$\begin{aligned} dy_1(t)/dt &= 3\,y_1(t) + 4\,y_2(t), \\ dy_2(t)/dt &= 4\,y_1(t) - 3\,y_2(t), \end{aligned}$$
   with initial conditions $y_1(0) = 1$, $y_2(0) = 0$.

   (a) Write down the system of ordinary differential equations in matrix–vector form. You should also give the initial conditions in vector form.

   (b) Find $\exp(At)$, where A is the matrix you found in (a).

   (c) Write down the solution of the system of differential equations.

3. Given that
$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
   Find $\exp(At)$. [Hint: work out the first few powers of A using matrix multiplication, matrices like A are called *nilpotent*.]

4. If B and C are square matrices, then prove the following properties:

   (a) If B and C commute, then $Be^{Ct} = e^{Ct}B$.

   (b) $e^{CBC^{-1}} = Ce^{B}C^{-1}$.

5. Suppose two units of mass are placed at $(0, 2, 2)$, one unit of mass at $(2, 1, 1)$, and one unit of mass at $(-2, 1, 1)$.

   (a) Find the moment of inertia tensor about the origin.

   (b) Find the principal axes and principal moments of inertia.

6. Text: §10.6 problem 18 (p. 513). Note that a quadratic form is said to be *positive definite* if it is greater than zero for all values of $\boldsymbol{x} \neq 0$. A quadratic form $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}$ is positive definite if and only if all eigenvalues of $\boldsymbol{A}$ are greater than zero.