MATH 3C : A2 Solutions

1. $v_1 = \frac{u_1}{\|u_1\|} = \frac{i}{\left[\int_0^\infty e^{-x}\,dx\right]^{1/2}} = i$

$v_2' = u_2 - \langle v_1, u_2\rangle v_1 = ix - \int_0^\infty (-i)(ix)e^{-x}\,dx \; i$

$= ix - i = i(x-1)$

$\Rightarrow v_2 = \frac{v_2'}{\|v_2'\|} = \frac{i(x-1)}{\left[\underbrace{\int_0^\infty (x-1)^2 e^{-x}\,dx}_{1}\right]^{1/2}} = i(x-1)$

$v_3' = u_3 - \langle v_1, u_3\rangle v_1 - \langle v_2, u_3\rangle v_2$

$= ix^2 - \int_0^\infty (-i)(ix^2)e^{-x}\,dx \; i - \int_0^\infty -i(x-1)(ix^2)\,e^{-x}\,dx \; i(x-1)$

$= ix^2 - 2i - 4i(x-1) = i(x^2 - 4x \; \cancel{+4-2} \; {+2})$

$\Rightarrow v_3 = \frac{v_3'}{\|v_3'\|} = \frac{i(x^2-4x+2)}{\left[\underbrace{\int_0^\infty (x^2-4x+2)^2 e^{-x}\,dx}\right]^{1/2}}$

$\left[\int_0^\infty (x^4 - 8x^3 + 20x^2 - 16x + 4)\,e^{-x}\,dx\right]^{1/2}$

$\left[\underbrace{24 - 48 + 40 - 16 + 4}_{2}\right]^{1/2}$

$\therefore \boxed{\{v_1, v_2, v_3\} = \left\{ i,\; i(x-1),\; \frac{i}{2}(x^2-4x+2) \right\}}$

2 (a) $v_1 = 1$

$$v_2 = u_2 - \frac{\langle v_1, u_2\rangle v_1}{\|v_1\|^2} = x - \frac{\cancel{\int_{-1}^{1} x\,dx}^{\,0}}{\|v_1\|^2} = x$$

$$v_3 = u_3 - \frac{\langle v_1, u_3\rangle v_1}{\|v_1\|^2} - \frac{\langle v_2, u_3\rangle v_2}{\|v_2\|^2}$$

$$= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx} - \frac{\cancel{\int_{-1}^{1} x^3\,dx}^{\,0}\, v_2}{\|v_2\|^2} = x^2 - 1/3$$

$$v_4 = u_4 - \frac{\langle v_1, u_4\rangle v_1}{\|v_1\|^2} - \frac{\langle v_2, u_4\rangle v_2}{\|v_2\|^2} - \frac{\langle v_3, u_4\rangle v_3}{\|v_3\|^2}$$

$$= x^3 - \frac{\cancel{\int_{-1}^{1} x^3\,dx}^{\,0}\, v_1}{\|v_1\|^2} - \frac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} x^2\,dx}\, x - \frac{\cancel{\int_{-1}^{1} (x^2 - 1/3)x^3\,dx}^{\,0}\, v_3}{\|v_3\|^2}$$

$$= x^3 - \frac{2}{5}\cdot\frac{3}{2}x$$

$$\therefore \boxed{\{v_1, v_2, v_3, v_4\} = \{1, x, x^2 - 1/3, x^3 - 3/5\,x\}}$$

(b) $v_i = C_{ij} u_j$, By inspection:

$$\underset{u \to v}{C} = \begin{bmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

write u_1 as L.C. of v's | u_2 as L.C. of v's | u_3 as L.C. of v's | u_4 as L.C. of v's

2. (c) $D\cdot 1 = 0$

$D\cdot x = 1$

$D(x^2 - 1/3) = 2x$

$D(x^3 - 3/5\,x) = 3x^2 - 3/5$

$$\boxed{D_v = \begin{bmatrix} 0 & 1 & 0 & 2/5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

(d)

$$C^{-1} = C_{v\to u} = \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_u = C^{-1} D_v C = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{matrix of } D = d/dx \text{ wrt } u \text{ basis } \{1,\, x,\, x^2,\, x^3\}}$$

$D_u = C^{-1}\ D_v\ C$

- C → transform from $u \to v$ basis
- D_v → do D in v basis
- C^{-1} → transform back from $v \to u$ basis
- D_u → D in u basis.

3(a) $\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix}$

Expand along 1^{ST} col. :

$$\to 1 \cdot \begin{vmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \\ 2 & 5 & 3 \end{vmatrix} = -\begin{vmatrix} -2 & 2 \\ 5 & 3 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ 2 & 5 \end{vmatrix}$$

$$= 16 + 4 = \boxed{20}$$

(b) (i) the Matrix of T wrt std basis

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \qquad T = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & -4 \\ 3 & -3 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \qquad \det(T) = 2\begin{vmatrix} 2 & -4 \\ -3 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix}$$

$$T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \qquad = 2(-10) + 9$$

$$= -20 + 9$$

$$= \boxed{-11}$$

4.(a) $|A-\lambda I| = 0 \to \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)\left[-(5+\lambda)(4-\lambda)+18\right] + 3\left[3(4-\lambda)-18\right]$

$+3\left[-18+6(5+\lambda)\right] = 0$

$= (1-\lambda)\left[\lambda^2+\lambda-2\right] \cancel{+} -9\cancel{(\lambda+2)} + \overset{9}{\cancel{18}}(\lambda+2) = 0$

$= (1-\lambda)(\lambda+2)(\lambda-1) + 9(\lambda+2) = 0$

$= \boxed{(\lambda+2)\left[-(\lambda-1)^2+9\right] = 0}$ char. polyⁿ.

(b) e-values : $\boxed{\lambda_1 = -2}$

$(\lambda-1)^2 = 9 \to (\lambda-1) = \pm 3$

$\to \boxed{\lambda_2 = 4\ ,\ \lambda_3 = -2}$

$\therefore$ e-values are $\boxed{4, -2, -2}$

$\lambda = 4 : \left[\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right]$

$\to \boxed{\xi = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}$ (dim of e-space assoc. to $\lambda = 4$ $= 1$)

(not nec. to normalize)

5. (10.2 problem 8)

assume $\{\lambda_j\}$ are distinct, ie. $\lambda_j \neq \lambda_k$ if $j \neq k$

Linear Independence of $\{x_j\}$ means that the only solution of

$$c_1 x_1 + \cdots + c_n x_n = 0 \quad \text{is} \quad c_1 = \cdots = c_n = 0$$

So, we need to show that all $c_j = 0$.

<u>Def</u>^{n}: if x_j is an e-vector of A with e-value λ_j then

$$(A - \lambda_j I) x_j = 0$$

$$\therefore \quad (A - \lambda_j I) c_j x_j = c_j (A - \lambda_j I) x_j = 0$$

similarly $\boxed{(A - \lambda_j I) c_k x_k = c_k (\lambda_k - \lambda_j) x_k}$

since $\boxed{A x_k = \lambda_k x_k}$. Successively apply $(A - \lambda_j I)$:

$$\therefore \left(\prod_{j=2}^{n} (A - \lambda_j I) \right) (c_1 x_1 + \cdots c_n x_n)$$

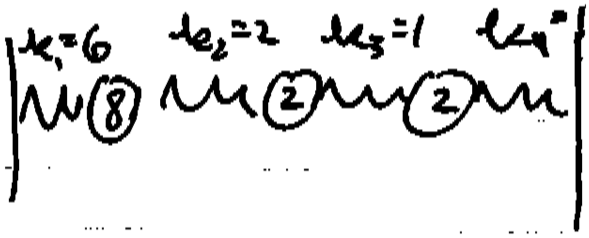
$$= c_1 \prod_{j=2}^{n} (\lambda_1 - \lambda_j) = 0 \quad \text{since } \lambda_j = \lambda_k \text{ if } j \neq k$$

$$\rightarrow c_1 = 0$$

can then do the same to show <u>all</u> $c_k = 0$ □

(~~eg.~~ ie. $\displaystyle \prod_{j \neq k}^{n} (A - \lambda_j I)(c_1 x_1 + \cdots + c_n x_n)$

$$= c_k \prod_{j \neq k}^{n} (\lambda_k - \lambda_j) = 0 \Rightarrow c_k = 0.)$$



6.

15. Potential ~~ener~~ energy is $\boxed{\frac{1}{2} k l^2 \text{ where } l \text{ is length of spring}}$

$$V = \frac{k_1}{2} x_1^2 + \frac{k_2}{2}(x_2 - x_1)^2 + \frac{k_3}{2}(x_3 - x_2)^2 + \frac{k_4}{2} x_3^2$$

(see p. 475)

$$= 3x_1^2 + (x_2 - x_1)^2 + \frac{1}{2}(x_3 - x_2)^2 + \frac{1}{2} x_3^2$$

~~the~~ Equations of motion are $\boxed{m\ddot{x} = -\frac{\partial V}{\partial x}}$

$$\rightarrow \quad m_1 \ddot{x}_1 = -6x_1 + 2(x_2 - x_1) = -8x_1 + 2x_2$$
$$m_2 \ddot{x}_2 = -2(x_2 - x_1) + (x_3 - x_2) = 2x_1 + x_3 - 3x_2$$
$$m_3 \ddot{x}_3 = -(x_3 - x_2) - x_3 = -2x_3 + x_2$$

16. In matrix form: using $m_1 = 8$, $m_2 = m_3 = 2$

$$\rightarrow \quad \boxed{\ddot{x} = \begin{bmatrix} -1 & 1/4 & 0 \\ 1 & -3/2 & 1/2 \\ 0 & 1/2 & -1 \end{bmatrix} x} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

let $x = c\, e^{i\omega t} \rightarrow \ddot{x} = -\omega^2 c\, e^{i\omega t} = -\omega^2 x$

$$\rightarrow \quad -\omega^2 c = \begin{bmatrix} -1 & 1/4 & 0 \\ 1 & -3/2 & 1/2 \\ 0 & 1/2 & -1 \end{bmatrix} c \quad , \quad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

This is an e-value problem

$$(A - \lambda I)c = 0 \quad \text{with } \lambda = -\omega^2$$

The e-values are: $\lambda_1 = -2$, $\lambda_2 = -1$, $\lambda_3 = -\frac{1}{2}$

$$\rightarrow \boxed{\omega_1 = \sqrt{2}, \quad \omega_2 = 1, \quad \omega_3 = 1/\sqrt{2}}$$

are the 3 fundamental frequencies.

17. The normal modes have the form

$c_j \cos(\omega_j t)$ where c_j is the e-vector associated with ω_j.

∴ the normal modes are

$$\boxed{\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \cos\sqrt{2}\,t, \quad \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \cos t, \quad \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cos t/\sqrt{2}}$$

18. Lowest frequency normal mode is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cos t/\sqrt{2} \qquad (\omega_3 = 1/\sqrt{2})$$

From (15) Potential energy $V = 3x_1^2 + (x_2 - x_1)^2 + \frac{1}{2}(x_3 - x_2)^2 + \frac{1}{2}x_3^2$

there ~~$x_1 = 1,\ x_2 = 2,\ x_3 = 2$~~

$x_1 = \cos t/\sqrt{2}$, $x_2 = 2\cos t/\sqrt{2}$, $x_3 = 2\cos t/\sqrt{2}$

$$\rightarrow V = a^2 \cos^2 t/\sqrt{2}\left[3 + 1 + 0 + \frac{4}{2}\right] = 6a^2\cos^2 t/\sqrt{2}$$

Since $a\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\cos t/\sqrt{2}$ is a normal mode for any a

& the Kinetic energy = $\boxed{\frac{1}{2} m \dot{x}^2}$

$$= \frac{1}{2}\left[8\left(\frac{a}{\sqrt{2}}\sin t/\sqrt{2}\right)^2 + 2\cdot\left(\frac{2a}{\sqrt{2}}\sin t/\sqrt{2}\right)^2 + 2\left(\frac{2a}{\sqrt{2}}\sin t/\sqrt{2}\right)^2\right]$$

$$= \frac{a^2}{2}\cdot\frac{1}{2}\sin^2 t/\sqrt{2}\left[8 + 8 + 8\right]$$

$$\boxed{K = 6a^2 \sin^2 t/\sqrt{2}}$$

$\therefore$ Total energy $= V + K = 6a^2(\cos^2 t/\sqrt{2} + \sin^2 t/\sqrt{2})$

$= 6a^2 =$ const. ✓