A1 - Math 3C

1. (a) def^n does not include zero vector
→ does not form a subspace.

(b) check closure :
vector addition $2(u+v)(0) = 2u(0) + 2v(0)$
$= u(1) + v(1)$
$= (u+v)(1)$ ✓
scalar mult. $2(\lambda u)(0) = \lambda 2u(0) = \lambda u(1)$ ✓
& contains 0 vector → forms subspace

(c) not closed under scalar multiplication for negative scalars → does not form a subspace

(d) Does form a subspace : closed under scalar mult. & vector addition & contains zero vector

$$(u+v)(t) = u(t) + v(t) = u(1-t) + v(1-t)$$
$$= (u+v)(1-t) \checkmark$$
$$(\lambda u)(t) = \lambda(u(t)) = \lambda u(1-t) \checkmark$$

2. (a) consider $\lambda_1(1, 1+i, 1) + \lambda_2(0, i, 1) + \lambda_3(1, i, 0) =$

$$\Rightarrow \begin{cases} \lambda_1 + \lambda_3 = 0 & (1) \rightarrow \lambda_1 = -\lambda_3 \\ (1+i)\lambda_1 + i\lambda_2 + i\lambda_3 = 0 & (2) \\ \lambda_1 + \lambda_2 = 0 & (3) \rightarrow \lambda_1 = -\lambda_2 \end{cases} \rightarrow \lambda_2$$

so we have from (1) + (3)

$\lambda_1 = \lambda$, $\lambda_3 = -\lambda$, $\lambda_2 = -\lambda$ for some λ

sub in (2)

$\rightarrow (1-i)\lambda = 0 \Rightarrow \lambda = 0$

so $\lambda_1 = \lambda_2 = \lambda_3 = 0$ is only sol^n & vectors are L.I.

(b) need to check vectors span $\mathbb{C}^3$, i.e. that $\exists$ $\lambda_1, \lambda_2, \lambda_3$ s.t.

$\lambda_1(1, 1+i, 1) + \lambda_2(0, i, 1) + \lambda_3(1, i, 0) = (a, b, c)$ $a, b, c \in \mathbb{C}$

But 3 L.I. vectors automatically span a v.s. of dim 3!
$\therefore$ they form a basis.

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2(1-i) \\ 1+i & i & i & 1-i \\ 1 & 1 & 0 & 1+i \end{array}\right]$

$\xrightarrow[\;③\to③-1\;]{②\to②-(1+i)1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2(1-i) \\ 0 & i & -1 & -3-i \\ 0 & 1 & -1 & -1+3i \end{array}\right]$

$\xrightarrow{②\to②(-i)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2(1-i) \\ 0 & 1 & i & -1+3i \\ 0 & 1 & -1 & -1+3i \end{array}\right]$

$\xrightarrow[\cancel{①\to①+i②}]{③\to③-②} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2(1-i) \\ 0 & 1 & i & -1+3i \\ 0 & 0 & -(1+i) & 0 \end{array}\right]$

can read off answer:

$\lambda_3 = 0, \quad \lambda_2 = -1+3i, \quad \lambda_1 = 2(1-i)$

$$\therefore \boxed{\begin{pmatrix} 2(1-i) \\ 1-i \\ 1+i \end{pmatrix} = 2(1-i)\begin{pmatrix} 1 \\ 1+i \\ 1 \end{pmatrix} + (-1+3i)\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}}$$

3. We first show that the two functions $f_1(x)$ and $f_2(x)$ are linearly independent over the interval $-\infty < x < \infty$. The two conditions

$$\begin{aligned} 0c_1 - x^2c_2 &= 0 \qquad x < 0 \\ x^2c_1 + 0c_2 &= 0 \qquad x \geq 0 \end{aligned}$$

imply that $c_1 = c_2 = 0$, and so $f_1(x)$ and $f_2(x)$ are linearly indpependent over the interval $-\infty < x < \infty$. But $W = \begin{vmatrix} 0 & -x^2 \\ 0 & -2x \end{vmatrix}$ if $x < 0$ and $\begin{vmatrix} x^2 & 0 \\ 2x & 0 \end{vmatrix}$ if $x > 0$, and $W = 0$ for all x.

4. (a) Need to show only sol^n of
$$\lambda_0 P_0 + \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 = 0$$
is $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 0$.

① → $\lambda_0 - \frac{1}{2}\lambda_2 = 0$ (coeff of x^0)

② $\lambda_1 - \frac{3}{2}\lambda_3 = 0$ (" " x^1)

③ $\frac{3}{2}\lambda_2 = 0$ (" " x^2)

④ $\frac{5}{2}\lambda_3 = 0$

→ $\lambda_2 = \lambda_3 = 0$ sub in ① → $\lambda_0 = 0$

sub in ② → $\lambda_1 = 0$

$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 0$ is only sol^n ✓

→ L.I.

space of cubic polynomials has dim = 4 (one basis is $1, x, x^2, x^3$) ∴ any set of 4 L.I. vectors in P_3 spans it.

∴ first 4 Legendre poly^n. spans it.

✓

(b) Linear ~~operator~~ operator must commute with scalar ~~add~~ mult. & vector addition:

ie need to show:

① $A(p+q) = A(p) + A(q)$

② $A(\lambda p) = \lambda A(p)$ for scalar λ.

$$①: A(p+q) = -(p+q) + 2\frac{d}{dx}(p+q)$$
$$= -p + 2\frac{dp}{dx} - q + 2\frac{dq}{dx}$$
$$= A(p) + A(q) \checkmark$$

(2) $A(\lambda p) = -(\lambda p) + 2\frac{d}{dx}(\lambda p)$

$= -\lambda p + \lambda\, 2\frac{dp}{dx} = \lambda\left(-p + 2\frac{dp}{dx}\right)$

$= \lambda A(p)$ ✓.

(c) Apply A to Legendre Basis & represent result in terms of Legendre Basis.

$AP_0 = -1 = -P_0$, first col. $= (-1, 0, 0, 0)$

$AP_1 = -x + 2 = 2P_0 - P_1$, 2^{nd} col. $= (2, -1, 0, 0)$

$AP_2 = -\frac{1}{2}(3x^2 - 1) + 6x = -P_2 + 6P_1$, 3^{rd} col $= (0, 6, -1, 0)$

$AP_3 = -\frac{1}{2}(5x^3 - 3x) + 15x^2 - 3$

$= -P_3 + 10P_2 + 2P_0$

$\Rightarrow$ 4^{th} col $= (2, 0, 10, -1)$

$$\Rightarrow \boxed{A = \begin{bmatrix} -1 & 2 & 0 & 2 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & -1 \end{bmatrix}}$$

(d) First, write u in terms of P_i:

$u = a P_0 + b P_1 + \left(\frac{2}{3}P_2 + \frac{1}{3}P_0\right)c$

$+ \left(\frac{2}{5}P_3 + \frac{3}{5}P_1\right)d$

$= \left(a + \frac{1}{3}c,\ + b + \frac{3}{5}d,\ + \frac{2}{3}c,\ \frac{2}{5}d\right)$ in terms of Legendre basis.

$$\therefore Au = \begin{bmatrix} -1 & 2 & 0 & 2 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a + \frac{1}{3}c \\ b + \frac{3}{5}d \\ \frac{2}{3}c \\ \frac{2}{5}d \end{bmatrix}$$

$$= \begin{bmatrix} -a - \frac{1}{3}c + 2b + 2d \\ -b + 4c - \frac{3}{5}d \\ -\frac{2}{3}c + 4d \\ -\frac{2}{5}d \end{bmatrix}$$

#5

We need to determine if $c_1\,\mathbf{I} + c_2\,\boldsymbol{\sigma}_x + c_3\,\boldsymbol{\sigma}_y + c_4\,\boldsymbol{\sigma}_z = \mathbf{0}$ only if $c_1 = c_2 = c_3 = c_4 = 0$. This matrix equation translates into the algebraic equation

$$\begin{aligned} c_1 \qquad\qquad + c_4 &= 0 \\ c_2 - ic_3 \qquad &= 0 \\ c_2 + ic_3 \qquad &= 0 \\ c_1 \qquad\qquad - c_4 &= 0 \end{aligned}$$

The corresponding determinant is

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{vmatrix} = -4i \neq 0$$

The fact that there is a nonzero determinant says that only the solution $c_1 = c_2 = c_3 = c_4 = 0$ exists, and so the matrices are linearly independent.

6.(a) Need to check that solutions are
1. closed under vector addition
2. " " scalar mult.
(3. contain zero vector)

3 : $y=0$ is a solution ✓

2 : if y is a sol^n check that λy is also a sol^n:

$$\sum_{j=0}^{n} a_j(x)(\lambda y)^{(j)} = \sum_{j=0}^{n} a_j(x)\lambda y^{(j)}$$

scalar

$$= \lambda \underbrace{\sum_{j=0}^{n} a_j(x) y^{(j)}}_{=0 \text{ since } y \text{ is a sol}^n} = 0 \quad ✓$$

1 if y_1 & y_2 are sol^n's, check that y_1+y_2 is a sol^n:

$$\sum_{j=0}^{n} a_j(x)(y_1+y_2)^{(j)} = \sum_{j=0}^{n} a_j(x)(y_1^{(j)}+y_2^{(j)})$$

$$= \underbrace{\sum_{j=0}^{n} a_j(x) y_1^{(j)}}_{=0 \text{ since } y_1 \text{ is a sol}^n} + \underbrace{\sum_{j=0}^{n} a_j(x) y_2^{(j)}}_{=0 \text{ since } y_2 \text{ is a sol}^n}$$

$= 0$ ✓

∴ sol^n's of hom. linear ODE form a subspace.

6.(b) check for closure under vector addition
i.e. if y_1 & y_2 are sol^n's of eqn:

$$y_1'' + y_1 y_1' + 3y_1 = 0$$
$$y_2'' + y_2 y_2' + 3y_2 = 0$$

is $y_1 + y_2$ a sol^n?

sub $y = y_1 + y_2$ into eqn:

$$(y_1+y_2)'' + (y_1+y_2)(y_1+y_2)' + 3(y_1+y_2) = 0$$

$$\rightarrow y_1'' + y_2'' + (y_1+y_2)(y_1'+y_2') + 3y_1 + 3y_2 = 0$$

$$\rightarrow \underbrace{y_1'' + y_1 y_1' + 3y_1}_{=0} + \underbrace{y_2'' + y_2 y_2' + 3y_2}_{=0}$$

$$+ \underbrace{y_1 y_2' + y_1' y_2}_{\text{this does not nec. } =0} = 0$$

$\therefore$ sol^n's of nonlinear ODE do <u>not</u> form a subspace.