Topology change of vortices using stochastic differential equations

N. K.-R. Kevlahan

Department of Mathematics & Statistics

McMaster University, Hamilton

Canada

Abstract

We propose two semi-analytic models for the interaction of N three-dimensional vortex filaments. These models allow for topology change of the vortices, e.g. reconnection. Both models are stochastic differential equations where the effect of diffusion is modelled via a Gaussian white noise forcing of the inviscid equations. The vorticity distribution is the ensemble average of many realizations, each of which contains N vortices. The first model is a straightforward extension of the semi-inviscid asymptotic approximation of Klein et al. [1] for nearly parallel vortices, while the second may be used for vortex filaments of arbitrary geometry.

Key words: Vortex filament, merging, reconnection, stochastic differential equation.

PACS: 47.32.Cc, 47.27.Eq

Email address: `kevlahan@mcmaster.ca` (N. K.-R. Kevlahan).

URL: `www.math.mcmaster.ca/kevla` (N. K.-R. Kevlahan).

1 Introduction

There is no simple, numerically efficient and accurate model of all stages of vortex reconnection. This is because viscous diffusion is necessary for full vortex reconnection, and viscosity renders line vortex models, with their relatively simple Hamiltonian dynamics, inapplicable. Despite this, many semi-inviscid models have been developed that include a simple approximation for core dynamics, e.g. [1–3]. These models fail once the vortices approach within a core radius, which leads to the development of a singularity in curvature (called a *hairpin* or *kink*). Various more or less *ad hoc* ways of dealing with reconnection have been proposed [4,5]. These methods use a physically-based algorithm to give the end result of the reconnection. Obviously, the intermediate stages of the reconnection are not resolved.

In this paper we propose two numerically efficient models for all stages of the interaction of N vortex filaments, including reconnection. The first model, described in §2.1, is a simple stochastic extension of the asymptotic approximation of Klein et al. [1] for nearly parallel vortices. A more general model, for vortices of arbitrary geometry, is presented in §2.2. Because they are not straightforward asymptotic approximations of the incompressible Navier–Stokes equations, the accuracy of these stochastic models will be tested by comparing them with full direct numerical simulations (DNS) of vortex interaction and reconnection. We hope the models presented here will help us better understand vortex filament interaction, especially at high Reynolds number.

2 Stochastic models for vortex reconnection

2.1 Nearly parallel vortices

The simplest model for the interaction of three-dimensional vortex filaments was derived by Klein et al. [1]. They consider the case of nearly parallel vortices (e.g. nearly aligned with the $z-$axis), and assume that the perturbation amplitudes are much smaller than the perturbation wavelengths, which are also much larger than the core radius. As usual in such semi-inviscid theories, they also assume that the separation between vortices is much larger than the core radius. With these assumptions the interaction between vortex filaments is approximated by two-dimensional point vortex interaction in planes perpendicular to the $z-$axis, while the self-interaction is given by a geometrically simple form of the local induction approximation. These equations are remarkably simple, and are easy to solve numerically to high precision. The numerical solutions for the interaction of perturbed nearly parallel filaments presented in [1] show that the equations develop a kink where the filaments are closest. At this point the theory breaks down and the solution is singular.

We propose to extend the theory of Klein et al. to include viscosity by adding Gaussian white noise forcing to the filament equations. We obtain the following stochastic partial differential equation (SDE):

$$\frac{\partial \mathbf{X}_j}{\partial \tau} = J\left[\Gamma_j \frac{\partial^2 \mathbf{X}_j}{\partial z^2}\right] + J\left[\sum_{k \neq j}^{N} 2\Gamma_k \frac{(\mathbf{X}_j - \mathbf{X}_k)}{|\mathbf{X}_j - \mathbf{X}_k|^2}\right] + \sqrt{2\nu'}\mathbf{b}_j \tag{1}$$

where $\mathbf{X}_j(z,\tau) = (x_j(z,\tau), y_j(z,\tau))$ are the coordinates of the vortex centrelines, Γ_j are their circulations, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{b}_j(z,\tau)$ are independent

Gaussian random variables with mean zero and variance one. Time has been re-scaled by 4π so $\nu' = 4\pi\nu$.

Formally, the only change with respect to the model of Klein et al. is the addition of the Gaussian white noise forcing. However, the interpretation of the solution is very different. The filament position $\mathbf{X}_j(t)$ is now a random variable, and the vorticity distribution is given by its probability density function (PDF). The position of the vortex centreline is the mean of $\mathbf{X}_j(t)$. We will see that the SDE model remains non-singular well beyond the time when the model of Klein et al. fails. This gives suggests that a model of this type could reproduce some aspects of vortex reconnection.

When $N = 2$ we can use the complex notation of [1] to derive the following equations for the interaction of a pair of filaments

$$\frac{\partial \psi_1}{\partial \tau} = i\frac{\partial^2 \psi_1}{\partial z^2} + 2i\Gamma\frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'}b_1 \tag{2}$$

$$\frac{\partial \psi_2}{\partial \tau} = i\frac{\partial^2 \psi_2}{\partial z^2} - 2i\frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'}b_2 \tag{3}$$

where $\psi_j = x_j(z,\tau) + i\,y_j(z,\tau)$, $b_j(z,t) = b_{j1} + i\,b_{j2}$, we have set $\Gamma_1 = 1$, $\Gamma = \Gamma_2/\Gamma_1$.

Finally, in the case of two dimensions the linear self induction term is zero, and we obtain the following simple equations for the interaction of a pair of two-dimensional vortices

$$\frac{\partial \psi_1}{\partial \tau} = 2i\Gamma\frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'}b_1$$

$$\frac{\partial \psi_2}{\partial \tau} = -2i\frac{\psi_1 - \psi_2}{|\psi_1 - \psi_2|^2} + \sqrt{2\nu'}b_2$$

The special case of two-dimensions and two identical vortices ($N = 2$, $\Gamma =$

1) was proposed as a model for vortex merger by Agullo & Verga [6]. Note that because the curvature term vanishes in two dimensions there is no self-interaction in this case.

In each case the initial condition is singular: a vortex line in three dimensions, or a point vortex in two dimensions. We can construct a non-singular initial condition by simply setting to zero all terms except the stochastic term until the core has the desired thickness.

Because it does not include any self-interaction, the dynamics of the two-dimensional SDE model are both qualitatively and quantitatively incorrect (compare figures 1 a and b). Nevertheless, the model does eventually produce a single Gaussian vortex. A simple correction which includes self-interaction by using Gaussian vortices at the non-SDE point vortex positions to advect the point vortices in the SDE model gives qualitatively correct results (see figure 1c).

The three-dimensional model (2-3) appears to perform better, probably because some self-interaction is included via the curvature term (i.e. the local induction approximation). Until the singular time $\tau_* \approx 0.52$ the SDE model closely matches the semi-inviscid model, except in regions where the vortices are close. For $\tau > \tau_*$ the SDE model continues to be well-behaved, although the solution eventually becomes inaccurate. A comparison of the direct numerical simulation solution of Marshall et al. [7] with the SDE model suggests that in fact the main source of error is in fact the assumption of nearly parallel vortices, which is not justified during the reconnection.

(a) (b) (c)

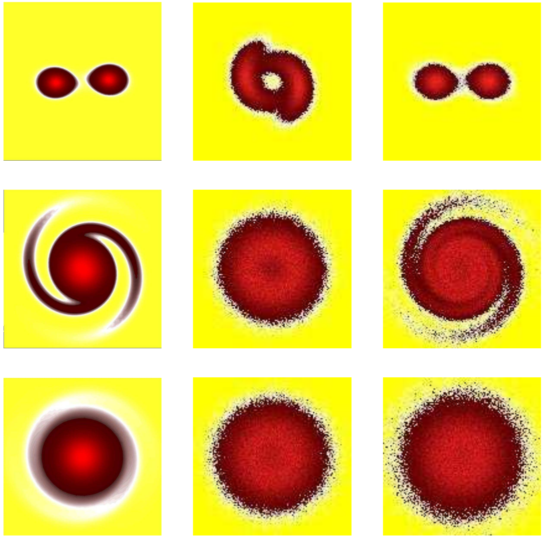

Fig. 1. Two-dimensional vortex merger, time increases from top to bottom. (a) Exact solution. (b) SDE model. (c) SDE model corrected to include self-interaction.

2.2 General SDE model

We now generalize the model for the viscous interaction of nearly parallel vortex filaments presented in the previous section. In order to treat the interaction of vortices which are not nearly parallel, we use the full Biot–Savart integral for a line vortex to calculate the effect of other vortices on the given filament, and include self-interaction via the general local induction approximation. With these modifications the deformation of the vortices is not limited, although vortex length is approximately conserved in any realization. The stochastic term must also be modified. We include diffusion only in the radial direction, and thus the Gaussian noise forcing is applied in the plane normal to the local tangent vector of the vortex. The geometry for this model is shown in figure 2.

With these assumption the general SDE model for the interaction of N vortices

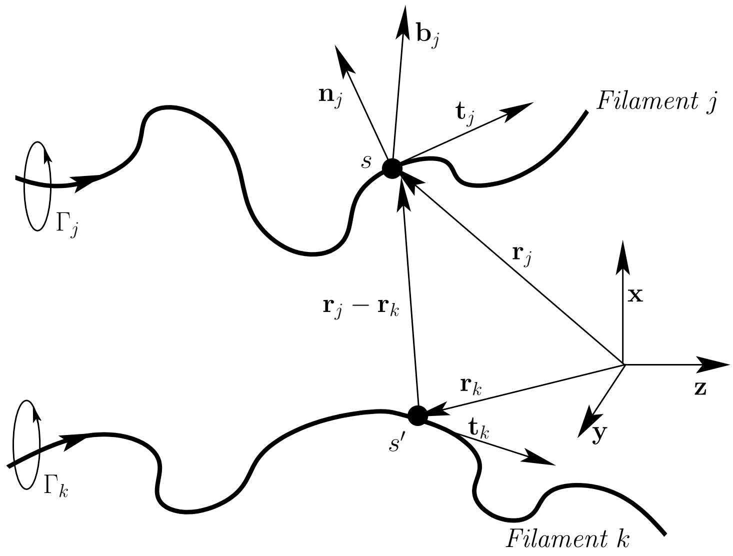

Fig. 2. Filament geometry.

becomes

$$\frac{\partial \mathbf{r}_j(s)}{\partial \tau} = -\sum_{k \neq j}^{N} \Gamma_k \int_{C_k} \frac{(\mathbf{r}_j(s) - \mathbf{r}_k(s')) \times \, \mathrm{d}\mathbf{t}_k(s')}{|\mathbf{r}_j(s) - \mathbf{r}_k(s')|^3} + \Gamma_j \kappa_j(s) \mathbf{b}_j(s) + \sqrt{2\nu'}(b_1 \mathbf{b}_j(s) + b_2 \mathbf{n}_j(s)). \tag{4}$$

where κ is the curvature and b_1 and b_2 are independent Gaussian random variables (generated separately for each position s and time τ).

We will present detailed comparisons of vortex reconnection calculated using this model with full DNS and the nearly parallel vortex SDE model presented above.

References

[1] R. Klein, A. J. Majda, K. Damodaran, Simplified equations for the interactions of nearly parallel vortex filaments, J. Fluid Mech. 288 (1995) 201–248.

[2] L. Ting, R. Klein, Viscous vortical flows, Springer, 1991.

[3] R. Klein, O. Knio, L. Ting, Representation of core dynamics in slender vortex filament simulations, Phys. Fluids 8 (1996) 2415.

[4] A. Chorin, Hairpin removal in vortex interactions, J. Comput. Phys. 91 (1990) 1–21.

[5] D. Kivotides, A. Leonard, Computational model of vortex reconnection, Europhys. Lett. 63 (3) (2003) 354–360.

[6] O. Agullo, A. D. Verga, Exact two vortices solution of Navier–Stokes equations, Phys. Rev. Lett. 78 (1997) 2361–2364.

[7] J. Marshall, P. Brancher, A. Giovannini, Interation of unequal anti-parallel vortex tubes, J. Fluid Mech. 446 (2001) 229–252.