Flow through cylinder arrays calculated using d'Arcy penalisation

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Abstract

The penalisation method introduced by [1], [4] to represent solid boundaries without modifying the computational grid is implemented in the pseudo-spectral method for the two-dimensional Navier–Stokes equations in vorticity-stream function formulation. The method is based physically on the d'Arcy equations for a medium of variable porosity, and has been shown mathematically to converge to the correct solution with a well-controlled error. We consider two types of periodic cylinder arrays: a square array with flow parallel to the array (inline array), and the same array with flow at -45 degrees to the array (rotated array). Such arrays are similar to the tube bundles used in the cooling circuit of nuclear reactors. Drag and Strouhal frequency are measured and where possible compared to experimental results. The inline and rotated arrays are found to have very different properties, both quantitatively and qualitatively.

1 Introduction

In calculating fluid flows of engineering interest, one is often confronted with high Reynolds number flow around complex solid obstacles (e.g. high speed flow over an aircraft or through a turbine). The equations for such a flow are the usual Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{\rho}\nabla P - \frac{1}{Re}\Delta\boldsymbol{u} = \boldsymbol{F}, \qquad (1)$$

$$\nabla\cdot\boldsymbol{u} = 0, \qquad (2)$$

where $\boldsymbol{u}$, P, Re and $\boldsymbol{F}$ are respectively the velocity, pressure, Reynolds number and external forcing; together with the boundary conditions on the surface of the obstacle $\partial\Omega$,

$$\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \quad \boldsymbol{u}\cdot\boldsymbol{\tau}|_{\partial\Omega} = 0, \qquad (3)$$

i.e. normal and tangential components of velocity are zero at the solid boundary. Unfortunately, implementing such boundary conditions numerically normally requires a computational mesh adapted to the shape of the body. Such meshes can now be generated relatively easily, but they are poorly adapted to the fastest numerical methods, which generally work best on very simple grids (e.g. rectangular). This is not such a serious drawback for low speed flows, but high speed (high Reynolds number flows) contain such a large range of active length scales that computation on an irregular grid is impractical.

One way around this problem is to keep the optimal simple grid and instead change the equation being solved. This general approach is called *penalisation* because one adds an extra term to the homogeneous Navier–Stokes equations that penalises flow within the obstacles. A variety of such penalisation methods have been used in the past (e.g. [3]), but they have suffered from the fact that they have been based on *ad hoc* mechanical models, cannot provide an *a priori* estimate of the error (or even guarantee that the model always converges to the physical solution).

The penalisation method used here is based physically on a limit of the d'Arcy equation for a porous medium and, importantly, has been shown to converge to the true solution with a well-controlled error. The d'Arcy penalisation equations are

$$\frac{\partial \boldsymbol{u}_\epsilon}{\partial t} + \boldsymbol{u}_\epsilon\cdot\nabla\boldsymbol{u}_\epsilon + \frac{1}{\rho}\nabla P - \frac{1}{Re}\Delta\boldsymbol{u}_\epsilon = $$

$$= \boldsymbol{F} - \frac{1}{\epsilon}\chi_S(\mathbf{x},t)\boldsymbol{u}_\epsilon, \qquad (4)$$

$$\nabla\cdot\boldsymbol{u}_\epsilon = 0, \qquad (5)$$

where the last term on the right hand side is the penalisation, which is controlled by the parameter $\epsilon \ll 1$. The mask $\chi_S(\mathbf{x},t)$ is equal to 1 in the solid and 0 in the fluid, and specifies the location and shape of the obstacle. In the limit $\epsilon \to 0$ the error of the penalisation is given by [1]

$$||\boldsymbol{u} - \boldsymbol{u}_\epsilon|| \approx \epsilon^{3/4}. \qquad (6)$$

Using this formulation, the forces on the obstacle (e.g. drag and lift) is simply given by an integral of the penalisation term over the computational domain,

$$\boldsymbol{F} = \frac{1}{\epsilon}\int_\Omega \chi_s(\mathbf{x},t)\boldsymbol{u}_\epsilon(\mathbf{x},t)\,\mathrm{d}\mathbf{x}. \qquad (7)$$

In this paper we apply the d'Arcy penalisation method to two-dimensional moderate Reynolds number flow in the vorticity-streamfunction formulation,

$$\frac{\partial\omega}{\partial t} + \frac{\partial\omega}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\omega}{\partial y}\frac{\partial\psi}{\partial x} - \frac{1}{Re}\Delta\omega = \frac{1}{\epsilon}\nabla\cdot\left(\chi_S\nabla\psi\right), \quad (8)$$

where ω is the vorticity, ψ is the streamfunction, $\omega = -\Delta\psi$, Again, the last term on the right hand side is the d'Arcy penalisation. These equations are solved using a standard pseudo–spectral method [2] on a square grid with a maximum resolution of 288^2.

2 Results

The penalisation method discussed in the introduction has been applied to two periodic flows: inline (flow parallel to square cell) and rotated square (flow at -45 degrees to square cell) arrays of circular cylinders. In each case the periodic cell contains one cylinder and the spacing ratio (pitch to diameter) of the cylinders is $P/D = 1.5$.

Figure 1: (a) Strouhal frequency as a function of Reynolds number for inline (squares) and rotated square (diamonds) cylinder arrays. The shaded region indicates the range of values found experimentally by [5].

Strouhal frequency $St = fD/U_\infty$ (where f is the vortex shedding frequency, D is the tube diameter and U_∞ is the mean flow) as a function of Reynolds number is shown in figure 1. The numerical results are compared with experimental results from [5] for measurements at various rows in a five row array. Note that the agreement with experiment is reasonable up to about $Re = 500$, which suggests that Strouhal frequency is dominated by two-dimensional effects until this Reynolds number. The Strouhal frequency for the inline array is lower than that of the rotated array. This difference is even more pronounced when rms drag values are compared: at $Re = 500$ the rotated array has an rms drag of 6.12, while the drag of the inline array is only 1.63.

The forces and oscillation frequencies acting on the rotated square array are significantly higher than the inline array, even though the physical arrangement of the tubes is identical (only the direction of the mean flows is different). This difference is likely due to fact that in the rotated array since the mean flow is not aligned with an axis of the array it has no clear path, and thus the flow becomes effectively more turbulent.

3 Conclusions

Although the results presented here are promising, there are a number of areas that require further work. The standard pseudo-spectral method gives only linear reduction of the error with reduction of grid point spacing, and for this reason we have investigated the possibility of retaining the simplicity of the pseudo-spectral approach (e.g. periodic boundary conditions, no need to calculate pressure explicitly when solving Navier–Stokes in three dimensions) while improving the convergence of the error. This can be done by simply transforming the relevant finite difference operator (e.g. Laplacian) to Fourier space.

Another difficulty is that the problem is *stiff* in time because of the small penalisation parameter ϵ. This means that in any explicit time scheme the time step must be $O(\epsilon)$. Standard stiff schemes (e.g. Gear) are impractical for pseudo-spectral methods, but the recently introduced Krylov subspace approach [6] appears promising.

Further comparisons with experiment are necessary to fully explore the potential advantages and limitations of the method. One important question concerns the size of the periodic cell: is only one cylinder per cell enough to capture the physics?

The combination of the pseudo-spectral method and the penalisation allow for a straightforward extension to three dimensions, and work is underway on this problem. Finally, it should be recalled that the principal advantages of the penalisation approach described here are its simplicity and the fact that a rigorous asymptotic error estimate exists.

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