NONLINEAR INTERACTIONS IN TURBULENCE WITH STRONG IRROTATIONAL STRAINING

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Abstract

The rate of growth of the nonlinear terms in the vorticity equation are analysed for a turbulent flow with r.m.s. velocity u_0 and integral length scale L subjected to a strong uniform irrotational plane strain S, where $(u_0/L)/S = \epsilon \ll 1$. The Rapid Distortion Theory (RDT) solution is the zeroth order term of the perturbation series solution in terms of ϵ. We use the asymptotic form of the convolution integrals for the zeroth order nonlinear terms when $\exp(St) \gg 1$ to determine when (in wavenumber k and time t) the perturbation series in ϵ fails, and hence estimate precisely the domain of validity of inviscid and viscous RDT.

1 Introduction

In most turbulent flows the large scale velocity field is a straining motion, either irrotational or rotational, that changes slowly on the time scale of the turbulent eddies — for example flow over waves, turbulence entering engines etc. The main practical and fundamental question concerns how the statistical and eddy structure of the turbulence is distorted by the strain and how its other properties are changed, such as mixing caused by the separation of fluid elements, or dissipation caused by transfer of kinetic energy to small scales. A key problem of turbulence research is to study how the nonlinear interaction between eddies or between Fourier components are affected by distortion.

In a review by Hunt & Carruthers (1990) it was shown that many aspects of distorted turbulent flows are determined by the *linear* interaction between the turbulence and the large scale mean straining flow and for which the nonlinear interaction can be neglected. Rapid Distortion Theory (RDT) is the term used to describe these methods and detailed assumptions of this simplified, though where appropriate quite powerful, approach.

The only theoretical calculations of the nonlinear interaction for turbulent flows have been based on statistical physics concepts, such as Direct Interaction Approximation and EDQNM (reviewed recently by McComb 1990; Lesieur 1990). They have not shown how and when the nonlinear interaction dominates linear distortion effects, and therefore have not provided insight into the limitations of RDT.

In this paper we explore a different approach based on a general asymptotic analysis of the nonlinear terms (expressed in terms of convolution of Fourier transforms) for the fluctuating vorticity field; no assumptions are made about its initial form provided its amplitude is small compared with the mean strain. This allows us to calculate the next term in the expansion of which RDT is the zeroth order term, and hence to estimate accurately the period of validity of RDT.

The nonlinear terms in the equation for the fluctuating vorticity when the turbulence $\boldsymbol{\omega}$ is undergoing a large scale plane strain $\boldsymbol{U}(\mathbf{x})$ are those representing the vortex lines being randomly rotated and stretched (by the terms $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$) and advected (by the term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}$) by the small scale turbulence $\boldsymbol{u}$. The approach hitherto for estimating the nonlinear terms relative to the linear terms $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{U}$, has been to assume either that $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$ has the same value as in undistorted turbulence $(\sim (u(l)^2/l^2)_{t=0})$, in which case the nonlinear terms are always greater than any exponentially decreasing linear term, or that the magnitude of $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$ is determined by the maximum value of $\boldsymbol{\omega}$ and $\boldsymbol{u}$ in the distorted flow in which case $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$ could grow even *faster* than the fastest growing linear term. Neither of these is likely because, as seen in the numerical simulation, the eddy structure changes in such a way as to reduce the nonlinear vorticity distortion terms! In the most frequently used Reynolds stress models, these nonlinear terms are based on estimates of the local distorted value of the turbulence covariance components and not on the distorted eddy structure. However the latter point has been demonstrated as crucial in modelling the nonlinear rotating and stretching effect, through an idealised analysis of large scale and small scale turbulence undergoing distortion (Kida & Hunt 1989) and through a statistical analysis of Direct Numerical Simulations by Mansour, Shih & Reynolds (1991) using a similar tensorial representation of the distorted spectra.

This paper summarises the main results of a detailed analysis of the distortion of turbulence caused by a large scale irrotational plane straining flow; the full details of this investigation will appear in a later article. This type of distortion was chosen because of its importance in a number of engineering problems and because of its fundamental importance in developing the basic theory of turbulence structure. We make the interesting discovery that the RDT assumptions remain valid for a relatively long time, usually until the final period of viscous decay, provided that the average in the compressed direction of the initial turbulent velocity component in this direction is zero. This condition is satisfied, for example, if the flow is bounded in the compressed direction. Estimates of the validity time of RDT are made and compared to the classical results.

2 Rapid Distortion Theory

The conditions for RDT are usually defined for eddies of length scale l and velocity scale $u(l)$ undergoing some kind of distortion over a time T_D, where the distortion may be an imposed strain of strength S, the sudden introduction of a boundary (in the frame of reference moving with the mean flow) or body forces etc. RDT is stated to be valid *either* if T_D is so rapid that the nonlinear terms in the vorticity equation have a negligible effect on the vorticity of the eddy ($\sim u(l)/l$), on the relevant time scale $\tau(l)$ of the eddy $l/u(l)$, *or* if the linear effects on $\boldsymbol{\omega}$ of the distortion (e.g. $\sim S(u/l)$ for a straining distortion) are much stronger than the nonlinear self-induced straining by the turbulence ($\sim (u/l)^2$).

Taking the integral scale $l = L$ and r.m.s. velocity u_0 this leads to two possible conditions for RDT

$$t \ll \tau(L) \sim L/u_0(t), \tag{1}$$

$$\text{or} \quad S \gg u_0(t)/L. \tag{2}$$

The latter condition for the strength of the strain rate may be satisfied even when the distortion is applied over a long period i.e. for slowly changing turbulence, and *may* be a valid condition for the accurate use of RDT. Thus we may combine the above conditions to give $u_0(t)/L \ll \max(S, 1/t)$. Hence in any given context one must take care to define precisely the term 'RDT'.

As explained in the introductory section, these conditions have been derived from the vorticity equation by inspection, and by assuming that the nonlinear terms are of the same order as in the undistorted turbulence. However, where the turbulence is strongly distorted the nonlinear stretching term $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$ can be much less than the undistorted estimate of $(u(l)/l)^2$, a good example being where the distorted flow forms into strong straight vortex tubes where vorticity is greater than the applied strain rate S; in this case the nonlinear vortex stretching term vanishes identically.

The purpose of this paper is to refine these estimates for the validity of RDT by considering the RDT solution to be the zeroth order term in an asymptotic expansion for the distorted turbulence. The validity range of RDT (in time and wavenumber) may then be found by determining when this expansion breaks down. This method should lead to much more precise estimates for the validity of RDT and to an understanding of how the nonlinear terms are affected by irrotational straining.

3 Results

The magnitude of the nonlinear terms depends sensitively on the amplitude of the eddies with large length scales perpendicular to the direction x_1 of positive strain. If the average of the velocity component u_2 in the convergence direction x_2 is initially zero

$$\int_{-\infty}^{+\infty} u_2(\mathbf{x}) \, \mathrm{d}x_2 = 0, \tag{3}$$

then the zeroth order nonlinear terms always remain smaller than the linear terms, even those that decrease exponentially. In this case RDT fails at a relatively long time $t \sim L/u_0 k^{-3}$ independent of ϵ (where k is the wavenumber), and the maximum amplification of vorticity under RDT is $\omega_1/S \sim \epsilon \exp(\epsilon^{-1}) \gg 1$. If (3) does not apply, the zeroth order nonlinear terms increase faster than the linear terms by a factor $O(\exp(St))$. RDT then fails at a relatively short time $t \sim 1/S \ln(\epsilon^{-1}k^{-3})$, and the maximum amplification of vorticity under RDT is $\omega_1/S \sim 1$. The analytical results on the growth of the nonlinear terms have been confirmed by numerical evaluation of the integrals for a particular form of eddy.

Viscous effects dominate when $t \gg 1/S \ln(k^{-1}(Re/\epsilon)^{1/2})$ (where Re is the Reynolds number), and RDT fails immediately in this range.

Thus we find that the usual order of magnitude estimate for the time period of the validity of RDT, namely that $u_0(t)/L \ll \max(S, 1/t)$, is an underestimate since $u_0(t)/L$ increases exponentially in time. Expressed in similar terms, we find instead that $u_0/L \ll 1/t$ if (3) is satisfied, and $u_0/L \ll S/\exp(St)$ otherwise, where $u_0 = u_0(t = 0)$. Interestingly, the 'crudest' order of magnitude estimate, $u_0/L \ll 1/t$, is also the most accurate if (3) is satisfied (e.g. bounded flows)!

Perhaps the most general point to emerge is that a weak random vorticity field can be amplified by a larger scale strain so that the strain rate can become of the same order as that of the applied strain. This is because the nonlinear processes, which might have inhibited this growth, are themselves inhibited by the straining. In other words strained turbulence adjusts itself so as to reduce to a consistent extent the 'scrambling' effects on its own amplified vorticity. This helps explain why weak turbulence can be so strongly amplified at the stagnation point of cylinders (Sadeh & Brauer 1980).

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