TRACE MAPS IN ALGEBRAIC TOPOLOGY

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Let V be a finite dimensional vector space over a field $\mathbb{F}$ of characteristic zero. The *trace map* is the unique linear functional, trace : $\mathrm{Hom}_{\mathbb{F}}(V,V) \to \mathbb{F}$, such that for all endomorphisms S,T of V, $\mathrm{trace}(ST) = \mathrm{trace}(TS)$ and $\mathrm{trace}(\mathrm{id}) = \dim V$. The trace appears in algebraic topology in the well known Lefschetz Fixed Point Theorem, one version of which asserts:

Theorem. Let M be a smooth closed connected oriented manifold and $f : M \to M$ a smooth map. Then the intersection number of the graph of f with the diagonal of $M \times M$ is equal to the *Lefschetz number* of f, defined by

$$L(f) = \sum_i (-1)^i \,\mathrm{trace}(f_i : H_i(M;\mathbb{Q}) \to H_i(M;\mathbb{Q})).$$

The trace map can be more generally defined in the context of a symmetric monoidal category $\mathcal{C}$, i.e., a category equipped with a "tensor product" with suitable properties. If R is a monoid in $\mathcal{C}$ (generalizing the notion of a ring in the case $\mathcal{C}$ is the category of abelian groups) and N is a left R-module satisfying an appropriate finiteness condition then there is a trace morphism in a homotopy category associated to $\mathcal{C}$,

$$\mathrm{trace} : \mathrm{Hom}_R(N,N) \longrightarrow \mathcal{F}(\mathcal{C}\mathrm{HH}_\bullet(R;R)),$$

where the receptacle of the trace, $\mathcal{F}(\mathcal{C}\mathrm{HH}_\bullet(R;R))$, is a "Hochschild complex".

Applications to parametrized fixed point theory (the study of the fixed points of a fiber preserving map $f : E \to E$ where E is a fiber space $p : E \to B$) arise by taking $\mathcal{C}$ to be a suitable symmetric monoidal category of chain complexes or of symmetric spectra.

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