LAST (family) NAME: ________________________ Test # 1

FIRST (given) NAME: ________________________ Math 2MM3

ID # : ____________________ February 12, 2009

Tutorial # : ____ Instructors: Dr. J.-P. Gabardo
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Test duration: 1 **hour**

Instructions: You **must** use permanent ink. Tests submitted in pencil will not be considered later for remarking. This test consists of 8 problems on 12 pages (make sure you have all 12 pages). The last two pages are for scratch or overflow work. There is a formula sheet on page 12. The total number of points is 50. Do not add or remove pages from your test. No books, notes, or "cheat sheets" allowed. The only calculator permitted is the McMaster Standard Calculator, the Casio fx 991.
GOOD LUCK!

SOLUTIONS

#	Mark
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
TOTAL	

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NAME: ______________________ **ID #:**______________________

PART I: Multiple choice. Indicate your choice very clearly. There is only one correct answer in each multiple-choice problem. Circle the letter (a,b,c,d or e) corresponding to your choice. Ambiguous answers will be marked as wrong.

1. *(4 pts.)* Compute the length l of the curve C parametrized by the vector function

$$\mathbf{r}(t) = \langle 3\,\cos(t^2),\, 3\,\sin(t^2),\, 2\,t^3\rangle, \quad 0 \le t \le \sqrt{8}.$$

(a) $l = 50$

$\rightarrow$ **(b)** $l = 52$

(c) $l = 53$

(d) $l = 54$

(e) $l = 56$

Solution. We have

$$\mathbf{r}'(t) = \langle -6\,t\,\sin(t^2),\, 6\,t\,\cos(t^2),\, 6\,t^2\rangle$$

and

$$\|\mathbf{r}'(t)\| = \sqrt{6^2\,t^2\,\sin^2(t^2) + 6^2\,t^2\,\cos^2(t^2) + 6^2\,t^4} = 6\,\sqrt{t^2 + t^4} = 6\,t\,\sqrt{1+t^2}.$$

Therefore,

$$l = \int_0^{\sqrt{8}} \|\mathbf{r}'(t)\|\,dt = \int_0^{\sqrt{8}} 6\,t\,\sqrt{1+t^2}\,dt = \left[2\,(1+t^2)^{3/2}\right]_0^{\sqrt{8}} = 2\,(27-1) = 52.$$

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NAME: ______________________ **ID #:**______________________

2. *(4 pts.)* Consider the vector field

$$\mathbf{F}(x,y,z) = x\,y\,\mathbf{i} + x\,z^2\,\mathbf{j} + y^2\,z\,\mathbf{k}.$$

Compute $a = \|(\nabla \times \mathbf{F})(P)\|$ where $P = (1,2,2)$.

(a) $a = 6$

(b) $a = \sqrt{5}$

(c) $a = \sqrt{13}$

(d) $\rightarrow$ $a = 5$

(e) $a = 8$

Solution. We have

$$(\nabla \times F)(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\,y & x\,z^2 & y^2\,z \end{vmatrix} = (2\,y\,z - 2\,x\,z)\,\mathbf{i} + 0\,\mathbf{j} + (z^2 - x)\,\mathbf{k}$$

and

$$(\nabla \times F)(1,2,2) = \langle 4,\ 0,\ 3\rangle.$$

Hence

$$\|(\nabla \times F)(0,2,2)\| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

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NAME: ______________________ **ID #:**______________________

3. *(4 pts.)* A surface S contains two curves C_1 and C_2, which intersect at the point $(1, 1, 1)$. These curves are parametrized respectively by vector functions $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ defined by

$$\mathbf{r}_1(t) = \langle t^2,\, t,\, t^3\rangle, \qquad \mathbf{r}_2(s) = \langle e^s,\, 1-s,\, \cos s\rangle, \quad -\infty < t,\, s < \infty.$$

What is the equation of the plane tangent to S at $(1, 1, 1)$?

(a) $2\,x + y + 2\,z = 5$

(b) $x - y + 3\,z = 2$

→ **(c)** $x + y - z = 1$

(d) $x + 3\,y - 2\,z = 2$

(e) $3\,x + 2\,y - z = 4$

Solution. We have $\mathbf{r}_1'(t) = \langle 2\,t,\, 1,\, 3\,t^2\rangle$, and, since $\mathbf{r}_1(1) = \langle 1,\, 1,\, 1\rangle$, the tangent vector to the curve C_1 at the point $(1, 1, 1)$ is

$$\mathbf{r}_1'(1) = \langle 2,\, 1,\, 3\rangle.$$

Similarly, $\mathbf{r}_2'(s) = \langle e^s,\, -1,\, -\sin s\rangle$, and, since $\mathbf{r}_2(0) = \langle 1,\, 1,\, 1\rangle$, the tangent vector to the curve C_2 at the point $(1, 1, 1)$ is

$$\mathbf{r}_2'(0) = \langle 1,\, -1,\, 0\rangle.$$

Since both these tangent vectors are also tangent to the surface S at $(1, 1, 1)$, it follows that the vector $\langle 2,\, 1,\, 3\rangle \times \langle 1,\, -1,\, 0\rangle$ is orthogonal to S at $(1, 1, 1)$. We have

$$\langle 2,\, 1,\, 3\rangle \times \langle 1,\, -1,\, 0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix} = \langle 3,\, 3,\, -3\rangle$$

and the equation of the tangent plane to S at $(1, 1, 1)$ is thus $3\,(x-1) + 3\,(y-1) - 3\,(z-1) = 0$ or

$$x + y - z = 1.$$

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NAME: ______________________ **ID #:**______________________

4. *(4 pts.)* Compute the volume V of the parallelepiped generated by the vectors $\mathbf{u_1}$, $\mathbf{u_2}$, $\mathbf{u_3}$, where

$$\mathbf{u_1} = \langle 1,\ 3,\ 1 \rangle, \quad \mathbf{u_2} = \langle 2,\ 0,\ 2 \rangle, \quad \mathbf{u_3} = \langle -1,\ 1,\ -2 \rangle.$$

(a) $V = 4$

(b) $V = 5$

→ **(c)** $V = 6$

(d) $V = 7$

(e) $V = 8$

Solution. We have

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & -2 \end{vmatrix} = 6$$

Hence, $V = |6| = 6$.

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NAME: ______________________ **ID #:**______________________

5. *(4 pts.)* Assume that $w = w(x, y)$ is a differentiable function and that $x = \frac{u}{v}$ and $y = u^2 + v^2$. Suppose also that

$$\frac{\partial w}{\partial x}(1, 2) = 3, \quad \frac{\partial w}{\partial y}(1, 2) = -2.$$

Compute $T = \left.\frac{\partial w}{\partial u}\right|_{(u,v)=(-1,\,-1)}$.

(a) $T = -2$

(b) $T = -1$

(c) $T = 0$

$\rightarrow$ **(d)** $T = 1$

(e) $T = 2$

Solution. Since $x = (-1)\,(-1) = 1$ and $y = (-1)^2 + (-1)^2 = 2$ when $(u, v) = (-1,\,-1)$, it follows, from the chain rule, that

$$\left.\frac{\partial w}{\partial u}\right|_{(u,v)=(-1,\,-1)} = \frac{\partial w}{\partial x}(1, 2)\,\frac{\partial x}{\partial u}(-1, -1) + \frac{\partial w}{\partial y}(1, 2)\,\frac{\partial y}{\partial u}(-1, -1)$$

We have

$$\frac{\partial x}{\partial u}(-1, -1) = \left.\frac{1}{v}\right|_{(u,v)=(-1,\,-1)} = -1$$

and

$$\frac{\partial y}{\partial u}(-1, -1) = 2\,u|_{(u,v)=(-1,\,-1)} = -2.$$

Therefore,

$$T = 3\,(-1) + (-2)\,(-2) = 1$$

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NAME: ______________________ **ID #:**______________________

Part II: Provide all details and fully justify your answer in order to receive credit.

6. *(10 pts.)* Find the point of intersection of the plane tangent to the sphere $x^2+y^2+z^2=6$ at the point $(1,1,2)$ and the normal line to the ellipsoid $2\,x^2+3\,y^2+4\,z^2=5$ at the point $(1,-1,0)$.

Solution. The sphere is the level surface $F(x,y,z)=6$ where $F(x,y,x)=x^2+y^2+z^2$. We have $\nabla F(x,y,z)=\langle 2\,x,\ 2\,y,\ 2\,z\rangle$ and a normal vector to the sphere at the point $(1,1,2)$ is the vector $\nabla F(1,1,2)=\langle 2,\ 2,\ 4\rangle$.
The equation of the plane tangent to the sphere at $(1,1,2)$ is thus

$$2\,(x-1)+2\,(y-1)+4\,(z-2)=0 \quad \text{or} \quad x+y+2\,z=6.$$

Similarly, the ellipsoid is the level surface $G(x,y,z)=5$ where $G(x,y,x)=2\,x^2+3\,y^2+4\,z^2$. We have $\nabla G(x,y,z)=\langle 4\,x,\ 6\,y,\ 8\,z\rangle$ and a normal vector to the ellipsoid at the point $(1,-1,0)$ is the vector $\nabla F(1,-1,0)=\langle 4,\ -6,\ 0\rangle$.
The normal line to the elllipsoid at the point $(1,-1,0)$ has thus vector equation

$$\mathbf{r}(t)=\langle 1,-1,0\rangle+t\,\langle 4,\ -6,\ 0\rangle=\langle 1+4\,t,\ -1-6\,t,\ 0\rangle, \quad -\infty<t<\infty.$$

The plane tangent to the sphere and the normal line to the ellipsoid intersect when

$$(1+4\,t)+(-1-6\,t)+2\,(0)=6 \quad \text{or} \quad t=-3.$$

The point of intersection is thus

$$\boxed{P=(-11,\ 17,\ 0)}.$$

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NAME: ______________________ **ID #:**______________________

7. Suppose that the position of a particle at time t is described by the vector function

$$\mathbf{r}(t) = \langle \cos t,\, \sin t,\, e^t \rangle, \quad -\infty < t < \infty.$$

(a) (8 pts.) Compute the acceleration vector $\mathbf{a}(t)$ as well as the tangential and normal components of the acceleration, $a_T(t)$ and $a_N(t)$, and the curvature $\kappa(t)$, at any time t.

Solution. We compute

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -\sin t,\, \cos t,\, e^t \rangle, .$$

and

$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle -\cos t,\, -\sin t,\, e^t \rangle.$$

We have

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + (e^t)^2} = \sqrt{1 + e^{2t}},$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \sin t\, \cos t - \sin t\, \cos t + e^t\, e^t = e^{2t},$$

and

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & e^t \\ -\cos t & -\sin t & e^t \end{vmatrix} = e^t(\cos t + \sin t)\,\mathbf{i} + e^t(-\cos t + \sin t)\,\mathbf{j} + \mathbf{k}.$$

Therefore,

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{e^{2t}\,(\cos t + \sin t)^2 + e^{2t}\,(-\cos t + \sin t)^2 + 1} = \sqrt{1 + 2\,e^{2t}}.$$

We have thus

$$a_T(t) = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|} = \frac{e^{2t}}{\sqrt{1 + e^{2t}}},$$

$$a_N(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{1 + 2\,e^{2t}}}{\sqrt{1 + e^{2t}}},$$

and

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{1 + 2\,e^{2t}}}{(1 + e^{2t})^{3/2}}.$$

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NAME: ______________________ **ID #:**______________________

(b) (4 pts.) Verify (for $\mathbf{r}(t)$ as above) that $a_T^2(t) + a_N^2(t) = \|\mathbf{a}(t)\|^2$ for any t.

Solution. We have $\|\mathbf{a}(t)\|^2 = \cos^2 t + \sin^2 t + e^{2t} = 1 + e^{2t}$.
Thus,

$$\begin{aligned} a_T^2(t) + a_N^2(t) &= \left(\frac{e^{2t}}{\sqrt{1+e^{2t}}}\right)^2 + \left(\frac{\sqrt{1+2\,e^{2t}}}{\sqrt{1+e^{2t}}}\right)^2 \\ &= \frac{1+2\,e^{2t}+e^{4t}}{1+e^{2t}} \\ &= \frac{(1+e^{2t})^2}{1+e^{2t}} = 1 + e^{2t} = \|\mathbf{a}(t)\|^2. \end{aligned}$$

Alternatively, we have

$$\mathbf{a}(t) = a_T\,\mathbf{T} + a_N\,\mathbf{N},$$

and the vectors $\mathbf{T}$ and $\mathbf{N}$ are unit vectors orthogonal to each other.
Thus,

$$\begin{aligned} \|\mathbf{a}(t)\|^2 = \mathbf{a}(t)\cdot\mathbf{a}(t) &= (a_T\,\mathbf{T} + a_N\,\mathbf{N})\cdot(a_T\,\mathbf{T} + a_N\,\mathbf{N}) \\ &= a_T^2\,\|\mathbf{T}\|^2 + a_N^2\,\|\mathbf{N}\|^2 + 2\,a_T\,a_N\,\mathbf{T}\cdot\mathbf{N} \\ &= a_T^2(t) + a_N^2(t) \end{aligned}$$

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NAME: ____________________ **ID #:**____________________

8. (8 pts.) Consider the function $F(x, y, z) = x^2 + z^2 + z\,e^y$. Find a point $(x_0,\, y_0,\, z_0)$ on the level surface $F(x, y, z) = 4$ with the property that $F(x, y, z)$ increases the fastest at $(x_0,\, y_0,\, z_0)$ in the direction of the unit vector $\mathbf{u} = \langle \frac{2}{\sqrt{5}},\, 0,\, \frac{1}{\sqrt{5}} \rangle$.

Solution. A function $F(x, y, z)$ increases the fastest at a point P in the direction of $\nabla F(P)$. If $F(x, y, z) = x^2 + z^2 + z\,e^y$, we have

$$\nabla F(x, y, z) = \langle 2\,x,\; z\,e^y,\; 2\,z + e^y \rangle.$$

It follows that the vector $\nabla F(x_0, y_0, z_0)$ must have the same direction as the vector $\mathbf{u} = \langle \frac{2}{\sqrt{5}},\, 0,\, \frac{1}{\sqrt{5}} \rangle$ or the vector $\langle 2,\, 0,\, 1 \rangle$. There exists thus $\lambda > 0$ such that

$$\langle 2\,x_0,\; z_0\,e^{y_0},\; 2\,z_0 + e^{y_0} \rangle = \lambda\,\langle 2,\, 0,\, 1 \rangle.$$

This yields $x_0 = \lambda$, $z_0\,e^{y_0} = 0$ and $2\,z_0 + e^{y_0} = \lambda$.
Since $e^{y_0} \neq 0$, the equation $z_0\,e^{y_0} = 0$ yields $z_0 = 0$, which then implies that $x_0 = e^{y_0}$.
Since the point $(x_0,\, y_0,\, z_0)$ is on the level surface $x^2 + z^2 + z\,e^y = 4$. it follows that $x_0^2 = 4$ so $x_0 = 2$ (since $x_0 = \lambda > 0$) and $y_0 = \ln x_0 = \ln 2$. We have thus

$$\boxed{(x_0,\, y_0,\, z_0) = (2,\, \ln 2,\, 0)}\,.$$

NAME: ______________________ **ID #:**______________________

SCRATCH

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NAME: ______________________ **ID #:**______________________

Some formulas you may use:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}, \quad \kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}, \qquad a_N = \kappa v^2 = \frac{\|\mathbf{v} \times \mathbf{a}\|}{v} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$$

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\, \mathbf{b}$$

SCRATCH

THE END