Solution of Problem set #2

(Solution by F. Su)

Solution 1 *Page 94.* #2

For $\forall z \in \mathbb{C}\backslash\tilde{\gamma}$, γ *is a* C^1 *curve. Therefore the set* $\tilde{\gamma}$ *is closed. So,* $\exists h_0 > 0$, *such that* $D(z, 2h_0) \cap \tilde{\gamma} = \emptyset$. *Take* $h \in \mathbb{C}$ *and consider the limit* $h \to 0$ *of the difference quotient*

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \oint_\gamma \frac{1}{h}(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z})d\zeta \\ &= \oint_\gamma \frac{1}{(\zeta - z - h)(\zeta - z)}d\zeta\end{aligned}$$

For $\forall h < h_0$,

$$|\frac{1}{\zeta - z}| > \frac{1}{h_0}, \qquad |\frac{1}{\zeta - z - h}| > \frac{1}{h_0}$$

so, $\frac{1}{(\zeta-z)(\zeta-z-h)}$ *converges to* $\frac{1}{(\zeta-z)^2}$ *uniformly with respective to h. Hence*

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \oint_\gamma \lim_{h\to 0}\frac{1}{(\zeta - z - h)(\zeta - z)}d\zeta \\ &= \oint_\gamma \frac{1}{(\zeta - z)^2}d\zeta\end{aligned}$$

This shows that

$$\frac{\partial f}{\partial z} = \oint_\gamma \partial(\frac{1}{\zeta - z})d\zeta$$

Also, we have

$$\frac{\partial f}{\partial \bar{z}} = \oint_\gamma \partial_{\bar{z}}(\frac{1}{\zeta - z})d\zeta = 0$$

so f *is holomorphic on* $\mathbb{C}\backslash\tilde{\gamma}$

Consider an example when $z \in \tilde{\gamma}$*. In case* $\gamma(t) = t$*, let* $z = x + iy$

$$f(z) = \int_0^1 \frac{1}{t-z}dt = \frac{1}{2}\ln\frac{(1-x)^2+y^2}{x^2+y^2} + i(\arctan\frac{1-x}{y} + \arctan\frac{x}{y})$$

When $x_0 \in (0,1)$

$$\begin{aligned} \lim_{y\to 0+} f(z) &= \ln\frac{1-x_0}{y} + i\pi \\ \lim_{y\to 0-} f(z) &= \ln\frac{1-x_0}{y} - i\pi \end{aligned}$$

So, for any $x_0 \in (0,1)$*,* $f(z)$ *is not continuous (even not defined).*

Solution 2 *Page 94.* #9

Let

$$\begin{aligned} f(z) &= \sum a_k z^k \\ g(z) &= \sum b_k z^k. \end{aligned}$$

Since both series converge for $x \in (-1,1)$*, it follows that* f *and* g *are holomorphic in* $|z| < 1$*. Since* $f(\frac{1}{n}) = g(\frac{1}{n})$ *for* $n = 2, 3, \cdots$ *and the sequence has an accumulation point* $0 \in D(0,1)$*, then* $f(z) \equiv g(z)$*.*

By the uniqueness of Taylor series coefficients, $a_k = b_k$*, for* $\forall k$*.*

Solution 3 *Page 94.* #11

a). $R = 1$*, and series diverges for all points on* $|z| = 1$*.*

c).

$$\limsup_{k\to\infty} |(\log k)^{\log k}|^{\frac{1}{k}} = \limsup_{k\to\infty} |k^{\log\log k}|^{\frac{1}{k}} = \limsup_{k\to\infty} e^{\frac{\log k \log\log k}{k}} = 0$$

$R = 1$*, and series diverges for all points on* $|z| = 1$.

e). $R = \frac{1}{3}$*, and series diverges for all points on* $|z| = 1$.

g). $R = e$*, and series diverges for all points on* $|z| = 1$.

Solution 4 *Page 94.* #17

$$f(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

Since $f(z)$ *has two poles on the circle* $|z| = 1$*, the power series diverges outside of the unit disk.*

Solution 5 *Page 94.* #21

The disc of convergence of series is $\bar{D}(0,1) = \{z, |z| \leq 1\}$. *On* $\bar{D}(0,1)$*,*

$$|\sum_{j=n+1}^{m} 2^{-j} z^{2^j}| \leq \sum_{j=n+1}^{m} 2^{-j} \leq \frac{1}{2^n}$$

So, the series converges uniformly on $\bar{D}(0,1)$. *Thus,* $f(z)$ *is holomorphic* $D(0,1)$ *and continuous on* $\bar{D}(0,1)$ *and for any* $z \in D(0,1)$*,*

$$f'(z) = \sum_{j=0}^{\infty} z^{2^j - 1}$$

If w is a 2^Nth root of unity, then for $j \geq N$, $w^{2^j} = (w^{2^N})^{2^{j-N}} = 1$, and for any $\varepsilon > 0$ and $r = (1-\varepsilon)^{\frac{1}{2^m}}$, then $m \longrightarrow \infty$ as $r \longrightarrow 1$

$$\sum_{j=2^N}^{m} (rw)^{2^j} = \sum_{j=2^N}^{m} r^{2^j} > (m - 2^N + 1)(1-\varepsilon)$$

so,

$$\begin{aligned} r|f'(rw)| &> \sum_{j=m+1}^{\infty} r^{2^j} + (m - 2^N + 1)(1-\varepsilon) - |\sum_{j=0}^{2^N-1} (rw)^{2^j}| \\ &> (m - 2^N + 1)(1-\varepsilon) - \sum_{j=0}^{2^N-1} r^{2^j} \longrightarrow \infty \quad (\text{ as } r \longrightarrow 1) \end{aligned}$$

Thus, $f'(z)$ is unbounded around the point w. So $f'(z)$ is not holomorphic at the points w. Since the set $\cup_N\{2^N$th roots of unity $\}$ is dense on $\{|z| = 1\}$, so $f(z)$ can not extend to a holomorphic function on a larger disc center at $z = 0$ than $D(0,1)$.

Solution 6 *Page 147.* #1

Let $F(z) = \frac{f(z)}{R(z)}$. Then $F(z)$ is bounded around the points $P_1, P_2, \cdots, P_k$. By assumption, the zeros of $p(z)$ are also zeros of $f(z)$. $F(z)$ is also bounded around these points. So, the only singular points of $F(z)$ are removable singular points. This implys that $F(z)$ is an entire function, which is bounded. So, $F(z)$ must be constant, thus

$$f(z) = CR(z).$$

Solution 7 *Page 147.* #12 *Some examples are:*

1). $\sum_0^\infty \frac{1}{n^2} z^n + \sum_1^\infty \frac{1}{n^2 2^n z^n}$. *This series converges on* $\{\frac{1}{2} \leq |z| \leq 1\}$.

2). $\sum_0^\infty \frac{1}{n^2} z^n + \sum_1^\infty \frac{1}{2^n z^n}$. *This series converges on* $\{\frac{1}{2} < |z| \leq 1\}$.

3). $\sum_0^\infty z^n + \sum_1^\infty \frac{1}{2^n z^n}$. *This series converges on* $\{\frac{1}{2} < |z| < 1\}$.

Solution 8 *Page 147.* #15

a).

$$\lim_{z \longrightarrow P} f(z) = \infty$$
$$\lim_{z \longrightarrow P} \frac{1}{f(z)} = 0$$

b).

$$\frac{1}{f(z)} = \sum_{n=k+1} c_n (z-P)^n = (z-P)^k \sum_{n=k} c_n (z-P)^{n-k} = (z-P)^k g(z)$$

$g(z)$ is holomorphic in $D(P, r)$ and $g(P) \neq 0$, then $(z-P)^k f(z) = \frac{1}{g(z)}$ has removable singularity point at P.

c). If $(z-P)^k g(z)$ is bounded, then P is a removable singularity point of $(z-P)^k g(z)$, so

$$\begin{aligned} (z-P)^k g(z) &= \sum_{n=0}^{\infty} c_n (z-P)^n \\ g(z) &= \frac{1}{(z-P)^k} \sum_{n=0}^{\infty} c_n (z-P)^n \end{aligned}$$

and $g(z)$ is bounded, so at least, $\exists n < k$ such that $c_n \neq 0$. i.e. P is a pole of $g(z)$. We can see that the least such m is precisely the order of the pole.

Solution 9 *Page 147.* #24

$$\lim_{z \longrightarrow 0} z f(z) = \lim_{z \longrightarrow 0} \frac{z}{e^z - 1} = 1$$

$f(z)$ has simple pole at $z = 0$ and has Laurent expansion $\sum_{n=-1}^{\infty} c_n z^n$ with $c_{-1} = 1$

$$\begin{aligned} 1 &= \sum_{n=-1}^{\infty} c_n z^n \sum_{n=1} \frac{z^n}{n!} = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} c_{k-1} \frac{1}{(n+1-k)!}) z^n \\ c_n &= -\sum_{k=-1}^{n-1} c_k \frac{1}{(n+1-k)!} \end{aligned}$$

$c_0 = -\frac{1}{2}$, $c_1 = \frac{1}{12}$, $c_2 = 0$, $c_3 = -\frac{1}{6!}$, $c_4 = 0$.

Therefore the first several Bernoulli numbers are $B_1 = \frac{1}{6}$, $B_2 = 0$, $B_3 = -1$, $B_4 = 0$

Solution 10 *Page 147.* #27

a).

$$\lim_{z \longrightarrow 0} z \csc z = \lim_{z \longrightarrow 0} \frac{z}{\sin z} = 1$$

$\sin z = \sum_{n \geq 0} (-1)^n \frac{1}{(2n+1)!} z^{2n+1}$, *let* $\csc z = \sum_{n \geq -1} c_n z^n$, *then*

$$1 = \sum_{n \geq -1} c_n z^n \sum_{n \geq 0} (-1)^n \frac{1}{(2n+1)!} z^{2n+1}$$

$c_{-1} = 1,\ c_0 = 0,\ c_1 = \frac{1}{6},\ c_2 = 0,\ c_3 = \frac{7}{360}$

$$\csc z = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \cdots \qquad |z| > 0$$

c).

$$\frac{z}{(z-1)(z-3)(z-5)} = \frac{1}{2(z-1)}(\frac{5}{z-5} - \frac{3}{z-3})$$

$$= \begin{cases} \sum_{n=0}^{\infty}(\frac{3}{2^{n+2}} - \frac{5}{2^{2n+3}})(z-1)^{n-1} & |z-1| < 2 \\ -\frac{5}{2}\sum_{n=0}^{\infty}\frac{(z-1)^{n-1}}{4^{n+1}} - \frac{3}{2}\sum_{n=0}^{\infty}\frac{2^n}{(z-1)^{n+2}} & 2 < |z-1| < 4 \\ 5\sum_{n=0}^{\infty}\frac{2^{2n-1}}{(z-1)^{n+2}} - 3\sum_{n=0}^{\infty}\frac{2^{n-1}}{(z-1)^{n+2}} & |z-1| > 4 \end{cases}$$

$$= \begin{cases} \frac{1}{8}\frac{1}{z-1} + \frac{7}{32} + \frac{19}{128}(z-1) + \frac{43}{512}(z-1)^2 + \cdots & |z-1| < 2 \\ \cdots - \frac{3}{2}\frac{1}{(z-1)^2} - \frac{5}{8}\frac{1}{z-1} - \frac{5}{32} - \frac{5}{128}(z-1) - \cdots & 2 < |z-1| < 4 \\ \frac{1}{(z-1)^2} + \frac{7}{(z-1)^3} + \frac{34}{(z-1)^4} + \frac{148}{(z-1)^5} + \cdots & |z-1| > 4 \end{cases}$$

h).

$$\frac{e^z}{z^3} = \sum_{n=0}^{\infty}\frac{z^{n-3}}{n!} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6} + \cdots \qquad |z| > 0$$

i).

$$\frac{e^{\frac{1}{z}}}{z^3} = \sum_{n=0}^{\infty}\frac{1}{n!z^{n+3}} = \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{2z^5} + \frac{1}{6z^6} + \cdots \qquad |z| > 0$$