Section 5 Convergence properties of Fourier series

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Math 4FT
April 1, 2013

(i) Fejer's theorem.

A version of Dirichlet's theorem (Section 2 (v)) is as follows

Theorem 1: Suppose that $f \in C^r(\mathbb{T}^1)$ for $r \geq 1$. Then the partial sums for its Fourier series

$$S_n(f)(x) = \sum_{|k| \leq n} \hat{f}_k \frac{e^{ikx}}{\sqrt{2\pi}}$$

converge uniformly to $f(x)$, and the rate is given by

$$\sup_{x \in \mathbb{T}^1} |S_n(f)(x) - f(x)| := \| S_n(f) - f \|_\infty \leq \frac{\text{Const.}}{n^{r-1/2}}.$$

proof: We go back to using the Dirichlet kernel

$$D_n(x) = \sum_{|k| \leq n} \frac{1}{\sqrt{2\pi}} e^{ikx} = \frac{1}{\sqrt{2\pi}} \frac{\sin((n+\frac{1}{2})x)}{\sin(x/2)},$$

where recall that $\int_0^{2\pi} D_n(x)\,dx = 1$, and that

$$S_n(f)(x) = \int_0^{2\pi} D_n(x-y) f(y)\,dy.$$

The convergence question is the study of the difference

$$|S_n(f)(x) - S_m(f)(x)| = \Big| \sum_{n < |k| \leq m} \hat{f}_k \frac{e^{ikx}}{\sqrt{2\pi}} \Big| \qquad (m > n \text{ wlog})$$

$$\leq \sum_{n < |k| \leq m} \frac{1}{\sqrt{2\pi}} \frac{1}{|k|} |\hat{f'}_k| \qquad \text{since } f \in C^1$$

$$\leq \frac{1}{\sqrt{2\pi}} \Big(\sum_{n < |k| \leq m} \frac{1}{|k|^2} \Big)^{1/2} \Big(\sum_{n \leq k \leq m} |\hat{f'}_k|^2 \Big)^{1/2}$$

$$\leq \frac{C}{n^{1/2}} \| f' \| \qquad L^2 \text{ norm of } \frac{df}{dx}.$$

Hence uniform convergence for $f' \in C^1$, with a given rate.

NB Use problem 2 (iii) of homework 2 to deduce that the pointwise limit of the Cauchy sequences $\{S_n(f)(x)\}_{n=1}^{\infty}$ is indeed $f(x)$. ∎

However we can still be somewhat uncomfortable that we are asking that $f \in C^1$ to obtain convergence of $S_n(f)$ in C^0. Here is one direction of improvement on this situation.

Theorem 2 (Fejer's Theorem) Let $f \in C(\mathbb{T}^1)$ be a continuous 2π-periodic function. Then the arithmetic means of the partial sums $S_n(f)$ converge uniformly to $f(x)$.

proof: The arithmetic mean of n numbers $s_0, s_1, s_2 \cdots s_n$ is the sum

$$\frac{1}{n}(s_0 + s_2 + \cdots + s_{n-1}).$$

Therefore the arithmetic mean of the partial sums of a Fourier series is given by the expression.

$$\frac{1}{n}\left(S_0(f) + S_1(f) + \cdots + S_{n-1}(f) \right)$$

We should check that these averaged sums have a smoothing effect on the limit:

Study the case of limits that exist.

If $\lim_{n\to\infty} s_n = p$ then $\lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=0}^{n-1} s_j\right) = p$.

Question: are there sequences s_n which do not have limits, such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} s_j \quad \text{exists ?}$$

These are sometimes called "Cesaro sums".

Next step in the proof, derive an expression for the kernel of the arithmetic means of the Dirichlet kernels $D_n(x)$.

$$\frac{1}{n}\left[S_0(f) + S_1(f) + \dots + S_{n-1}(f)\right](x)$$

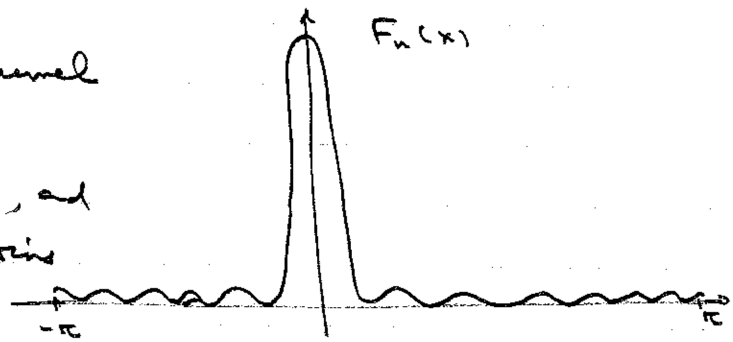
$$= \frac{1}{n}\sum_{j=0}^{n-1}\left(\int_0^{2\pi}\frac{1}{2\pi}\frac{\sin((j+\frac{1}{2})y)}{\sin(y/2)}f(x-y)\,dy\right)$$

$$= \int_0^{2\pi}\frac{1}{2\pi n}\left(\sum_{j=0}^{n-1}\frac{\sin((j+\frac{1}{2})y)}{\sin(y/2)}\right)f(x-y)\,dy$$

claim:

$$= \int_{-\pi}^{\pi}\underbrace{\frac{1}{2\pi n}\left(\frac{\sin(\frac{ny}{2})}{\sin(\frac{y}{2})}\right)^2}_{\text{the Fejer kernel.}} f(x-y)\,dy$$

Here's a picture of the kernel



In particular it is nonnegative, and does not rely upon cancellations in order to achieve a limit for large n.

proof of claim: $\sin((j+\frac{1}{2})x) = \operatorname{im}\left(e^{ijx}e^{ix/2}\right)$

$$\frac{1}{n}\sum_{j=0}^{n-1}\frac{\sin((j+\frac{1}{2})x)}{\sin(x/2)} = \frac{1}{n\sin(x/2)}\operatorname{im}\left(\sum_{j=0}^{n-1}e^{ijx}e^{ix/2}\right)$$

$$= \frac{1}{n\sin(x/2)}\operatorname{im}\left(\frac{1-e^{inx}}{1-e^{ix}}\cdot e^{ix/2}\right)$$

$$= \frac{1}{n\sin(x/2)}\operatorname{im}\left(-\frac{1-e^{inx}}{e^{ix/2}-e^{-ix/2}}\right)$$

$$= \frac{1}{n\sin(x/2)}\operatorname{im}\left[\frac{-1}{2i}\,\frac{1-e^{inx}}{\sin(x/2)}\right]$$

$$= \frac{1}{n \sin^2(x/2)} \left[\frac{1}{4} \left(e^{inx} - 2 + e^{-inx} \right) \right]$$

$$= \frac{1}{n} \frac{\sin^2(\frac{nx}{2})}{\sin^2(\frac{x}{2})} .$$

Continuation of the proof of Fejer's theorem: calculate the difference

$$\left[\frac{1}{n} \sum_{k=0}^{n-1} S_j(f) - f \right](x) = \int_{-\pi}^{\pi} F_n(y)\, f(x-y)\, dy - f(x)$$

$$= \int_{-\pi}^{\pi} F_n(y) \left(f(x-y) - f(x) \right) dy$$

To show that this is small for large n, divide into two parts.

(1) $|y| < \delta$ $\quad \left| \int_{-\delta}^{\delta} F_n(y) \left(f(x-y) - f(x) \right) dy \right|$

$$\le \int_{-\delta}^{\delta} F_n(y) \left| f(x-y) - f(x) \right| dy$$

(using that $F_n \geq 0$).

Take $\delta < \pi$ so small that $|f(x-y) - f(x)| < \varepsilon/2$ for all $|y| < \delta$; this can be done by uniform continuity.

$$\le \int_{-\delta}^{\delta} F_n(y)\, dy \times \frac{\varepsilon}{2}$$

$$\le \int_{-\pi}^{\pi} F_n(y)\, dy \times \frac{\varepsilon}{2} = \frac{\varepsilon}{2} .$$

(2) $\delta \le |y| \le \pi$:

$$\left| \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} F_n(y) \left(f(x-y) - f(x) \right) dy \right|$$

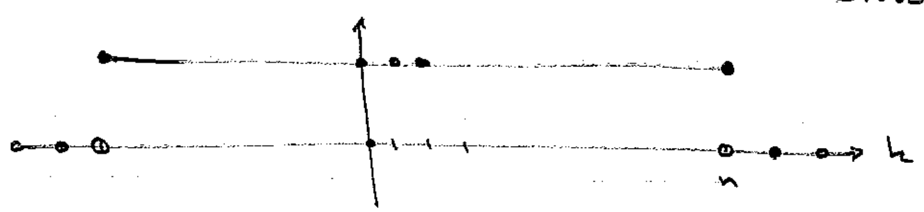
$$\le 2 \|f\|_\infty \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} F_n(y)\, dy \right)$$

$$= \frac{2}{n}\|f\|_\infty \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} \frac{\sin^2(\frac{ny}{2})}{\sin^2(y/2)} \, dy \right)$$

$$\le \frac{2}{n}\|f\|_\infty \frac{2\pi}{\sin(\delta/2)} .$$

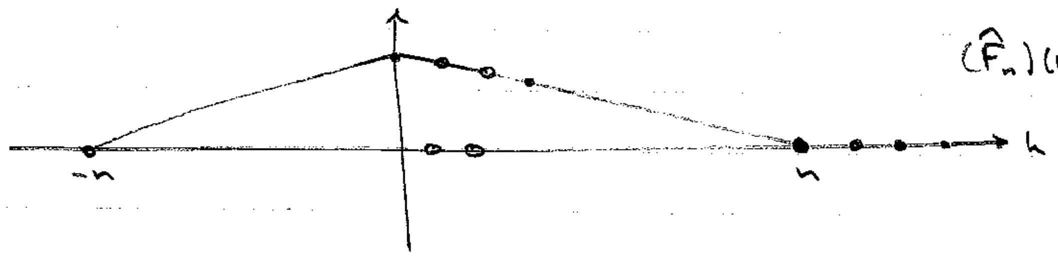
Fixing $\delta > 0$, we take n sufficiently large so that this two is smaller than $\varepsilon/2$. ∎

The shape of $\hat{D}_n(k)$ the Fourier transform of the Dirichlet kernel



$$\hat{D}_n(k) = \begin{cases} 1 & -n \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

The shape of the Fejer kernel $\hat{F}_n(k) =$



$$(\hat{F}_n)(k) = \begin{cases} \frac{n-|k|}{n} & 0 \le |k| \le n \\ 0 & \end{cases}$$

<u>Theorem 3</u> (Poisson formula) Let $f \in C(\mathbb{T}^1_\theta)$, then the harmonic extension of $f(\theta)$ to the disk $|x|^2 + |y|^2 \le 1$ is given by

$$u(r,\theta) = \sum_{k \in \mathbb{Z}} \hat{f}_k \frac{e^{ik\theta}}{\sqrt{2\pi}} r^{|k|} \quad \text{(in polar coords)}$$

This takes on its boundary data,

$$\lim_{r \nearrow 1} \left\| \left(\sum_{k \in \mathbb{Z}} \hat{f}_k r^{|k|} \frac{e^{ik\theta}}{\sqrt{2\pi}} \right) - f(\theta) \right\|_\infty = 0.$$

proof: Express the harmonic extension in terms of the Poisson kernel

$$u(r,\theta) = \sum_{k\in\mathbb{Z}} \hat{f}_k \frac{e^{ik\theta}}{\sqrt{2\pi}} r^{|k|}$$

$$= \int_{-\pi}^{\pi} \Big(\sum_{k\in\mathbb{Z}} \frac{e^{ik(\theta-\varphi)}}{2\pi} r^{|k|} \Big) f(\varphi)\, d\varphi$$

$$= \cancel{\tfrac{1}{2\pi}} \int_{-\pi}^{\pi} \underbrace{\frac{1}{2\pi} \frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}}_{P(r,\varphi-\theta)} f(\varphi)\, d\varphi .$$

For $|r|<1$ the Poisson kernel is positive. Hence use the same strategy for $|r|\to 1$ that was used for $n\to\infty$ in the Fejer kernel. ∎