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~~Th. Feb 14 2013~~

Math 4 FT

C3) The geometry of Hilbert space

Mon Feb 25, 2013

Fourier series is naturally associated with the space $L^2(\mathbb{T}^1)$ of square integrable periodic functions

$$L^2(\mathbb{T}^1) = \{ f(x) = f(x+2\pi) : \int_0^{2\pi} |f(x)|^2 dx < +\infty \}$$

To study this space of functions we will need
i) Lebesgue measure (defer for the moment; Royden background)
ii) the concept of a Hilbert space, an infinite dimensional analog of Euclidean space $\mathbb{C}^n$.

(i) Hilbert space in coordinates

First study Hilbert spaces w/o the extra complication of integrals and Lebesgue measure, and the question of completeness. We will study instead $\ell^2(\mathbb{Z})$.

Definition 1 The space $\ell^2(\mathbb{Z}) = \{ (z_n)_{n\in\mathbb{Z}}, z_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |z_n|^2 < +\infty \}$
: space of square summable sequences.

This subsection will prove that $\ell^2(\mathbb{Z})$ is a Hilbert space H.

Abstract definition of a Hilbert space.

(1) A set of points $z, w \in H$
- closed under addition

$$z + w \in H$$

and complex scalar multiplication

$$\alpha z \in H \quad \text{for any } \alpha \in \mathbb{C}$$

- A vector space.

ex: $z = (z_n)_{n\in\mathbb{Z}}$
$w = (w_n)_{n\in\mathbb{Z}}$
$z + w = (z_n + w_n)_{n\in\mathbb{Z}}$
$\alpha z = (\alpha z_n)_{n\in\mathbb{Z}}$

That is, satisfying the usual rules

$$z + w = w + z \qquad \text{and} \qquad \alpha(z+w) = \alpha z + \alpha w .$$

(2) Additionally, H has an inner product

$$(z, w) \in \mathbb{C} \qquad \text{ex: } \sum_{n\in\mathbb{Z}} z_n \overline{w_n}$$

obeying the rules

- $(z,w) = \overline{(w,z)}$
- $(z^{(1)} + z^{(2)}, w) = (z^{(1)}, w) + (z^{(2)}, w)$
- $\alpha\,(z,w) = (\alpha z, w) \overset{\text{implies}}{=} (z, \overline{\alpha} w) \qquad \alpha \in \mathbb{C}$

The inner product must be positive definite, nondegenerate

- $(z, z) \geq 0$ and $(z,z) = 0$ only when $z = 0$.

The points of H are considered a (metric) geometric space, where the distance between two points z, w is defined to be

$$\text{dist}\,(z,w) = \|z - w\| := \sqrt{(z-w, z-w)} .$$

Note that $\|z-w\| = 0$ only when $z = w$.

(3) Finally, we need that H be a complete space. That is, given a Cauchy sequence $\{z^{(m)}\}_{m=1}^{\infty} \subseteq H$,

$$\|z^{(m)} - z^{(n)}\| \to 0 \quad \text{as } m, n \to +\infty,$$

then there exists an actual limit point $z \in H$ such that

$$\|z^{(n)} - z\| \to 0 \quad \text{as } n \to \infty.$$

We say that the sequence $\{z^{(m)}\}$ converges to $z \in H$

Hilbert spaces can be finite dimensional, $\simeq \mathbb{C}^N$, or of countable infinite dimension (separable) such as $\ell^2(\mathbb{Z})$, or non-separable. We will mostly work with separable Hilbert spaces.

Note: $z = (z_1, z_2 \dots z_N) \in \mathbb{C}^N$, inner product $(z,w) = \sum_{j=1}^{N} z_j \overline{w_j}$.

Proposition 2 (Cauchy-Schwarz inequality) $|(z,w)| \le \|z\|\,\|w\|$.

proof: let $z, w \in H$, and consider $r \in \mathbb{R}$, $0 \le \theta < 2\pi$.
Then

$$z - re^{i\theta}w \in H$$

Its norm satisfies

$$0 \le \|z - re^{i\theta}w\|^2 = (z - re^{i\theta}w,\ z - re^{i\theta}w)$$

$$= (z,z) - (re^{i\theta}w, z) - (z, re^{i\theta}w) + (re^{i\theta}w, re^{i\theta}w)$$

$$= \|z\|^2 - 2\,\mathrm{re}\left[r e^{i\theta}(z,w)\right] + r^2\|w\|^2 .$$

Choose $\theta = \arg(z,w)$, to obtain

$$= \|z\|^2 - 2r\,|(z,w)| + r^2\|w\|^2$$

This is quadratic in r and non negative, hence the discriminant must be non-positive

$$4\left(|(z,w)|^2 - \|z\|^2\|w\|^2\right) \le 0$$ ☒

If equality holds, the discriminant vanishes. This implies that there is a zero, i.e. $\exists\, r$ such that

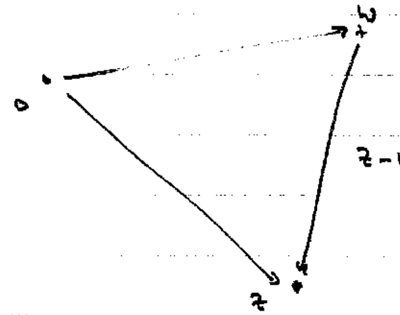
$$\|z - re^{i\theta}w\|^2 = 0 .$$

This means that $z = re^{i\theta}w = \alpha w$, $\alpha \in \mathbb{C}$. That is, the ~~points~~ z, w are (complex) proportional.

Corollary 3: The inner product is a continuous function on H.

proof. $|(z^{(1)},w) - (z^{(2)},w)| = |(z^{(1)} - z^{(2)}, w)| \le \|z^{(1)} - z^{(2)}\|\,\|w\|$,
therefore if $z^{(1)}$ approaches $z^{(2)}$, meaning $\|z^{(1)} - z^{(2)}\| \to 0$, then $(z^{(1)},w) \to (z^{(2)},w)$.

Proposition 4 (triangle inequality) $\|z-w\| \leq \|z\| + \|w\|$



proof: $\|z-w\|^2 = \|z\|^2 - 2\,\mathrm{re}(z,w) + \|w\|^2$
$\leq \|z\|^2 + 2\|z\|\|w\| + \|w\|^2$
$= (\|z\| + \|w\|)^2$ $\boxed{}$

You can also state this as $\|z+w\| \leq \|z\| + \|w\|$.

Define the angle θ between $z, w \in H$ to be
$\cos(\theta) := \frac{\mathrm{re}(z,w)}{\|z\|\|w\|}$, then $\|z+w\|^2 = \|z\|^2 - 2\|z\|\|w\|\cos\theta + \|w\|^2$

Hence if $(z,w) = 0$, then $\cos(\theta) = 0$, so $\theta = \frac{\pi}{2}$, a right angle.
Then

$$\|z+w\|^2 = \|z\|^2 + \|w\|^2$$ the Pythagorean theorem.

Theorem 5 The space $\ell^2(\mathbb{Z})$ is a Hilbert space.

proof: We have to check the above Hilbert space axioms.
Two points of $\ell^2(\mathbb{Z})$:
$$z = (z_n)_{n\in\mathbb{Z}}, \quad w = (w_n)_{n\in\mathbb{Z}}.$$
$$\sum_n |z_n|^2, \ \sum_n |w_n|^2 < +\infty.$$

(1) addition $z+w = (z_n+w_n)_{n\in\mathbb{Z}}$
$$\|z+w\|^2 = \sum_n |z_n+w_n|^2 \leq \sum_n |z_n|^2 + |w_n|^2 < +\infty$$
$$= \|z\|^2 + \|w\|^2$$

- scalar multiplication

$$\alpha z = (\alpha z_n)_{n\in\mathbb{Z}}, \quad \|\alpha z\|^2 = \sum_n |\alpha|^2 |z_n|^2 = |\alpha|^2\|z\|^2$$

- addition is commutative
- scalar multiplication distributes over addition.

(2) inner product $(z, w) := \sum_{n\in\mathbb{Z}} z_n \overline{w_n}$

check $= \overline{\sum_{n\in\mathbb{Z}} w_n \overline{z_n}} = \overline{(w, z)}$

respects addition and scalar multiplication

$$(z^{(1)} + z^{(2)}, w) = \sum_{n\in\mathbb{Z}} (z_n^{(1)} + z_n^{(2)})\overline{w_n} = (z^{(1)}, w) + (z^{(2)}, w)$$

$$\alpha(z, w) = \sum_{n\in\mathbb{Z}} (\alpha z_n)\overline{w_n} = \sum_{n\in\mathbb{Z}} z_n \overline{(\bar{\alpha} w_n)}$$

Positive nondegenerate

$$(z, z) = \sum_{n\in\mathbb{Z}} z_n \overline{z_n} = \sum_{n\in\mathbb{Z}} |z_n|^2 \geq 0$$

and $(z, z) = 0$ iff all $z_n = 0$.

(3) completeness of $\ell^2(\mathbb{Z})$.

The question is as do a Cauchy sequence $z^{(j)} = (z_n^{(j)})_{n\in\mathbb{Z}}$ has a $\ell^2(\mathbb{Z})$ limit.

proof of this fact:

A "sequence is Cauchy" means that $\forall \varepsilon > 0$ there exists a cutoff index J sufficiently large so that for all $j, n \geq J$ the $\| z^{(j)} - z^{(n)} \| < \varepsilon$.

For sequences, each element satisfies $|w_\ell|^2 \leq \sum_{n\in\mathbb{Z}} |w_n|^2$ therefore $\forall \varepsilon > 0$ there is J such that for $j, n \geq J$, the complex numbers $(z_\ell^{(j)})_{j=1}^\infty$ is a Cauchy sequence, Indeed

$$|z_\ell^{(j)} - z_\ell^{(n)}|^2 \leq \sum_{n\in\mathbb{Z}} |z_n^{(j)} - z_n^{(n)}|^2 = \| z^{(j)} - z^{(n)} \|^2 < \varepsilon^2$$

Hence each sequence $(z_\ell^{(j)})_{j=1,2,\ldots}$ has a limit z_ℓ to which it converges.

Claim: the sequence of limits $z := (z_\ell)_{\ell\in\mathbb{Z}}$ is the ℓ^2 limit of $(z^{(j)})_{j=1}^\infty$

Proof of this claim:

$$\lim_{n\to\infty} z_k^{(n)} = z_k$$

therefore $\lim_{n\to\infty} (z_k^{(j)} - z_k^{(n)}) = (z_k^{(j)} - z_k)$.

Given $\varepsilon > 0$, pick J sufficiently large that for all $j, n \geq J$,

$$\sum_k |z_k^{(j)} - z_k^{(n)}|^2 < \varepsilon^2 \qquad \left((z^{(j)})_{j=1,2} \text{ is Cauchy} \right)$$

Observe that $\forall$ cutoffs N

$$\lim_{n\to\infty} \sum_{|k|\leq N} |z_k^{(j)} - z_k^{(n)}|^2 = \sum_{|k|\leq N} |z_k^{(j)} - z_k|^2 .$$

therefore

$$\|z^{(j)} - z\|^2 = \lim_{N\to\infty} \sum_{|k|\leq N} |z_k^{(j)} - z_k|^2$$

$$= \lim_{N\to\infty} \left(\lim_{n\to\infty} \sum_{|k|\leq N} |z_k^{(j)} - z_k^{(n)}|^2 \right)$$

$$\leq \lim_{N\to\infty} \Big(\underbrace{\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} |z_k^{(j)} - z_k^{(n)}|^2}_{\text{in fact this is independent of } N} \Big)$$

$$\leq \varepsilon^2 .$$

Therefore $\|z^{(j)} - z\|^2 \leq \varepsilon^2$. Since $\varepsilon > 0$ is arbitrary, $z^{(j)}$ converges to z.

Incidentally, we must have $z \in \ell^2(\mathbb{Z})$ because

$$\|z\| \leq \|z^{(j)} - z\| + \|z^{(j)}\| \leq \|z^{(j)}\| + \varepsilon < +\infty,$$ by the triangle inequality. ∎

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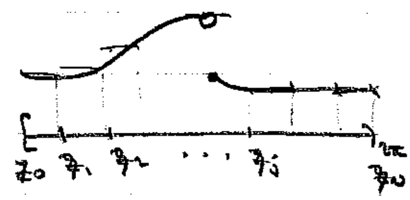
(ii) Lebesgue integral primer

Examine the Riemann integral (for flaws)

Given $f(x)$ defined on $[0, 2\pi)$, we want to make the integral

$$\int_0^{2\pi} f(x)\,dx \quad \text{as a limit.}$$



Divide $[0, 2\pi)$ into smaller intervals

$$z_0 = 0 < z_1 < z_2 \cdots < z_j < \cdots < z_N = 2\pi$$

$$0 < z_j - z_{j-1} \le \Delta .$$

Form the Riemann sums $\displaystyle\sum_{j=1}^{N} f_j \, (z_j - z_{j-1}) = \sum_{j=1}^{N} f_j \, \Delta_j$

where

$$f_j = f(z_j^*) \qquad \text{for some choice } z_j^* \in [z_{j-1}, z_j].$$

Possible choices:
- $z_j^* \in [z_{j-1}, z_j]$ arbitrary
- $f_j = \sup\limits_{z_{j-1} \le z \le z_{j}} (f(z))$
- $f_j = \inf\limits_{z_{j-1} \le z \le z_j} (f(x))$

<u>Definition 1</u> When the limit exists

$$\lim_{\Delta \to 0} \left(\sum_{j=1}^{N} f_j \, \Delta_{z_j} \right)$$

- for all possible partitions $\{z_j\}_{j=1}^{N}$
- $N \to \infty$
- all possible choices of f_j,

then the Riemann integral exists

$$\int_0^{2\pi} f(x)\,dx := \lim_{\Delta \to 0} \left(\sum_{j=1}^{N} f_j \, \Delta_{z_j} \right).$$

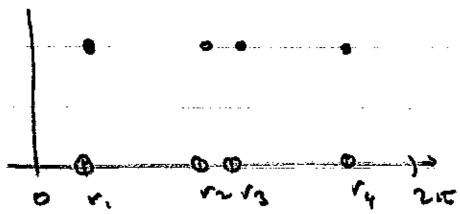
<u>Theorem 2</u> The Riemann integral of a continuous function on $[0, 2\pi]$ exists. The Riemann integral of a bounded function which is continuous except at finitely many points also exists.

<u>proof</u>: omitted.

example: Let $f_n(x) = \begin{cases} 1 & \text{for } x = r_1, r_2, \dots r_n \\ 0 & \text{otherwise} \end{cases}$,

where $r_1, r_2, \dots$ is a countable (dense) set R of points of $[0,2\pi)$.

Then



$$\int_0^{2\pi} f_n(x)\,dx = 0 .$$

A typical Riemann sum

$$\sum_{j=1}^{N} f_j \Delta_j = \sum_{\ell=1}^{n} 1\,\Delta_j \le n\Delta \to 0$$

$\begin{cases} 1 & \text{if } f_j = f(z_j) = f(r_\ell) \\ 0 & \text{otherwise.} \end{cases}$

However take the limit $\lim_{n\to\infty} f_n(x) := f(x) = \begin{cases} 1 & x \in R \\ 0 & x \notin R. \end{cases}$

Its Riemann sums are

$$0 = \sum_{j=1}^{N} \inf_{z_{j-1} \le z \le z_j} (f(z))\,\Delta_{z_j} < \sum_{j=1}^{N} \sup_{z_{j-1} \le z \le z_j} (f(z))\,\Delta_{z_j} = 2\pi$$

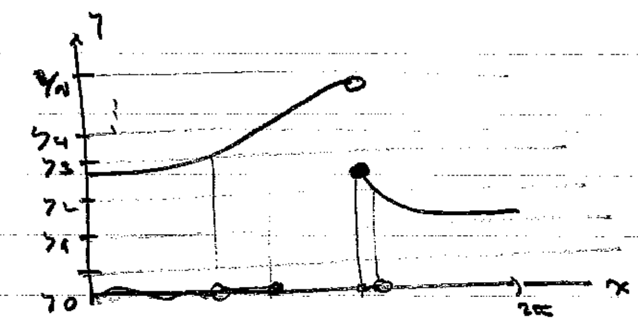
hence no limit, and the Riemann integral is not defined.

We will see that the Lebesgue integral of $f(x)$ is

$$\int_0^{2\pi} f(x)\,dx = \lim_{n\to\infty} \int_0^{2\pi} f_n(x)\,dx = 0 .$$

The Lebesgue integral:

Instead of dividing the x-axis into Δ sized units, divide instead the y-axis.



Approximate the integral $\int_0^{2\pi} f(x)\,dx$ by different sums

$$\sum_{j=1}^{N} y_j\, m(y_{j-1} < f(x) \le y_j),$$

and take the limit as $\Delta \to 0$ where $\Delta_j = y_j - y_{j-1} \le \Delta$.

When $f(x)$ is a continuous function, the sets
$\{x : y_{j-1} < f(x) \le y_j\}$ form a union of intervals (and pts)

What is their <u>measure</u> – a reasonable choice for intervals.

- Interval (a,b)

  $m(a,b) = b-a.$

- Open or closed intervals

  $m[a,b) = b-a = m(a,b] = m[a,b].$

- If $A = \bigcup_{j=1}^{\infty} I_j$ with $I_j = (a_j, b_j)$ nonoverlapping, then

  $$m(A) = \sum_{j=1}^{\infty} (b_j - a_j).$$

Desired properties of measure.

- If $A \subseteq B$ then $m(A) \le m(B)$.

- Hence, if A is a countable set, such as $A = \{r_j\}_{j=1}^{\infty} = \mathbb{R}$,
  set $B = (r_1 - \frac{\varepsilon}{2}, r_1 + \frac{\varepsilon}{2}) \cup (r_2 - \frac{\varepsilon}{4}, r_2 + \frac{\varepsilon}{4}) \cup (r_3 - \frac{\varepsilon}{4}, r_3 + \frac{\varepsilon}{4}) \cup \dots$

  $m(B) = \sum_{j=0}^{\infty} \frac{\varepsilon}{2^j} = 2\varepsilon$. Since $0 < \varepsilon$ arbitrarily small,

  $m(A) \le m(B) < 2\varepsilon \quad \forall \varepsilon \Rightarrow m(A) = 0.$

  Therefore we can conclude that $\int_0^{2\pi} f(x)\,dx = 0$
  when $f(x) = \begin{cases} 1 & x = r_j \\ 0 & \text{otherwise}, \end{cases}$

  and $0 = \lim_{n\to\infty} \int_0^{2\pi} f_n(x)\,dx = \int_0^{2\pi} \lim_{n\to\infty} (f_n(x))\,dx = \int_0^{2\pi} f(x)\,dx = 0$

Measurable sets:

- Intervals are measurable, $m(a,b) = b-a$
- Countable unions of disjoint intervals are measurable, $m(\bigcup_{j=1}^{\infty}(a_j,b_j)) = \sum_j (b_j - a_j).$
- we need more.

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Definition 3 The class of Borel sets $\mathcal{B}$ (subsets of $[0,2\pi)$)
(i) contains all (sub) intervals of $[0,2\pi)$
(ii) is closed under complements, and countable unions (and intersections)
(iii) is the smallest class of subsets satisfying (i) and (ii).

outline of start of Real 4 class

Definition 4 measure. If $E \in \mathcal{B}$ is a Borel set, define

$$m(E) = \inf_{\substack{I_j = (a_j, b_j) \\ E \subseteq \bigcup_j I_j}} \sum_j (b_j - a_j)$$

Proposition 5: (i) All open sets $O \subseteq [0,2\pi)$ are Borel sets.
proof: When $O \subseteq [0,2\pi)$, then $O = \bigcup_{j=1}^{\infty} (a_j, b_j)$, a countable union

(ii) All closed sets $G \subseteq [0,2\pi)$ are Borel sets.
proof: $G^c = O$ is open and hence is Borel.

ex 2 The Cantor set is Borel. (a subset of $[0,1]$ rather than $[0,2\pi)$)

C = the complement of the following union of disjoint open intervals.

$$C^c = (\tfrac{1}{3}, \tfrac{2}{3}) \cup \left[(\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9}) \right] \cup \cdots \in \mathcal{B}$$

This means $m([0,1] \setminus C) = \sum_{j=1}^{\infty} 2^{j-1} \left(\frac{1}{3^j}\right)$

$$= \frac{1}{3} \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j = \frac{1}{3}\left(\frac{1}{1-2/3}\right) = 1.$$

Therefore $m(C) = 0$.

There are still more sets than just Borel sets $\mathcal{B}$ which are measurable. Suppose that Z is a set such that

$$\inf_{Z \subseteq \bigcup_j I_j = \bigcup_j (a_j, b_j)} \sum_{j=1}^{\infty} (b_j - a_j) = 0, \quad \text{(a zero measure set)}.$$

Then $\mathcal{M}$:= the class of Lebesgue measurable sets $= \mathcal{B} \cup \mathcal{Z}$
where $\mathcal{Z}$ = all sets w. zero measure (defined as above).

NB: Not all sets are measurable.

Definition 6 A set $A \subseteq [0, 2\pi)$ is measurable $(A \in \mathcal{M})$ if $A = B \cup Z$
where $m(Z) = 0$ and $B \in \mathcal{B}$.
A (real valued) function is measurable if all of its semi-level sets are measurable sets.
$\{x \in [0, 2\pi) : f(x) < b\} \in \mathcal{M}$ for all $b \in \mathbb{R}$.

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$\Rightarrow$ examples: - $C[0, 2\pi)$ consists of Lebesgue measurable functions

- The sets $\{x : [0, 2\pi) : a \le f(x) < b\} \in \mathcal{M}$
$= \{x \in [0, 2\pi) : f(x) < b\} \cap (\{x \in [0, 2\pi) : f(x) < a\})^c$
- If $f(x) = \lim_{n\to\infty} f_n(x)$ is a pointwise limit of measurable functions $f_n(x)$, then $f(x)$ is measurable.

Definition 7 The Lebesgue integral of a nonnegative measurable function $f(x) \geq 0$ is defined as

$$\int_0^{2\pi} f(x)\,dx = \lim_{n\to\infty} \sum_{y_j} y_j\, m(\{x : y_{j-1}^{(n)} < f(x) \le y_j^{(n)}\})$$

where $|y_j^{(n)} - y_{j-1}^{(n)}| \to 0$ as $n \to \infty$.

Advantages of this definition:
(1) The Lebesgue integral of $f(x) \geq 0$, measurable will always exist (it may be $+$infinity).
(2) If $f(x)$ takes on both signs, define $f(x) = f_+(x) - f_-(x)$, where $f_+(x) = f(x) \vee 0$, Then define

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} f_+(x)\,dx - \int_0^{2\pi} f_-(x)\,dx$$
which is well defined as long as not both are infinite.

A measurable function $f(x)$ is integrable ($f \in L^1[0,2\pi)$) if both

$$\int_0^{2\pi} f_+(x)\,dx\ ,\quad \int_0^{2\pi} f_-(x)\,dx < +\infty\ ,$$

in which case

$$\int_0^{2\pi} |f(x)|\,dx = \int_0^{2\pi} f_+(x)\,dx + \int_0^{2\pi} f_-(x)\,dx := \|f\|_{L^1}.$$

Properties of measurable sets:

Proposition 8 Let $A, B \in \mathcal{M}$ the class of measurable sets.

(i) $0 \le m(A) \le m(B) \le 2\pi$ for $A \subseteq B \subseteq [0,2\pi)$

and $m(A) + m([0,2\pi)\setminus A) = 2\pi$

(ii) let $A_j \in \mathcal{M}$, $B_j \in \mathcal{M}$ $j = 1,2,3\ldots$, then

$$m\Big(\bigcup_{j=1}^{\infty} A_j\Big) \le \sum_{j=1}^{\infty} m(A_j)$$

and

$$m\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} m(A_j)$$ if disjoint; $A_j \cap A_\ell = \phi$ whenever $j \neq \ell$.

(iii) If $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_j \subseteq \ldots$ and $\bigcup_{j=1}^{\infty} B_j = A$, then

$$\lim_{j\to\infty} m(B_j) = m(A).$$

If $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_j \supseteq \ldots$ and $\bigcap_{j=1}^{\infty} B_j = A$ (and $m(B_1) < +\infty$) then

$$\lim_{j\to\infty} m(B_j) = m(A).$$

Properties of the Lebesgue integral:

Proposition 9: Let $f(x) \geq 0$, then

(i) $\int_0^{2\pi} \chi_A(x)\,dx = m(A)$, where χ_A is the indicator function of A.

(ii) If $\alpha \leq f(x)\,\chi_B(x) \leq \beta$

then $\alpha\, m(B) \leq \int_B f(x)\,dx \leq \beta\, m(B)$.

(iii) If $A \cap B = \emptyset$ then

$$\int_{A \cup B} f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx$$

(iv) $$\int_0^{2\pi} f_1 + f_2 \,dx = \int_0^{2\pi} f_1\,dx + \int_0^{2\pi} f_2\,dx .$$

Using these properties, several deeper results of Lebesgue theory are already available to us:

Theorem 10 (Chebyshev inequality) Let $f(x)$ be such that $\int_0^{2\pi} |f(x)|^2\,dx < +\infty$, then

$$m(\{x : |f(x)| \geq \lambda\}) \leq \frac{1}{\lambda^2}\int_0^{2\pi} |f(x)|^2\,dx$$

proof: Divide $[0,2\pi)$ into two sets; $A = \{x : |f(x)| \geq \lambda\}$ and $B = [0,2\pi)\setminus A$.
Then

$$\lambda^2 \chi_A(x) \leq |f(x)|^2 .$$

Therefore

$$\lambda^2 \int \chi_A\,dx \leq \int_A |f(x)|^2\,dx \leq \int_0^{2\pi} |f(x)|^2\,dx \qquad \blacksquare$$

Theorem 11 (Borel - Cantelli lemma) If the sets B_j, $j=1,2,\dots$ are measurable, and such that $\sum_{j=1}^{\infty} m(B_j) < +\infty$, then

$$m\Big(\bigcap_{n\geq 1} \bigcup_{j\geq n} B_j\Big) = 0$$

The interpretation of this statement - in combinatorial or probabilistic terms: the set $\bigcap_{n\geq 1} (\bigcup_{j\geq n} B_j)$ is the "set of points x that belong to infinitely many of the sets B_j".

<u>proof</u>: $m\left(\bigcap_{n\geq 1}\bigcup_{j\geq n} B_j\right) \leq m\left(\bigcup_{j\geq n} B_j\right) \leq \sum_{j\geq n} m(B_j)$.

This holds for any n, and since the sum on the RHS is finite, the tail of the sum must go to zero. Namely $\forall \varepsilon>0$, $\exists N(\varepsilon)$ such that for any $n\geq N(\varepsilon)$, $\sum_{j\geq n} m(B_j) < \varepsilon$. ☒

<u>Limit theorems</u>:

One of the biggest advantages of Lebesgue theory is that one understands the passage of limits under the integral sign.

<u>Sets</u>: Suppose $B_j \in \mathcal{M}$ for all $j=1,2,3\ldots$, then by definition $\bigcup_j B_j$ and $\bigcap_j B_j$ are also measurable.

<u>Functions</u>: Similarly, we could show that if $f_j(x)$, $j=1,2,3\ldots$ are measurable functions, then

$$f(x) = \inf_j (f_j(x)) \quad , \quad \sup_j (f_j(x)) \quad , \quad \underline{\lim}_{j\to\infty} (f_j(x)), \quad \overline{\lim}_{j\to\infty} (f_j(x))$$

are also measurable.

<u>Integrals</u>:

<u>Lemma 12</u> (Fatou's lemma) Suppose that all $f_j(x)\geq 0$ $j=1,2,3\ldots$ and that $\lim_{j\to\infty} f_j(x) = f(x)$ a pointwise limit. Then

$$\int_0^{2\pi} f(x)\,dx := \int_0^{2\pi} \lim_{j\to\infty} f_j(x)\,dx \leq \underline{\lim}_{j\to\infty} \int_0^{2\pi} f_j(x)\,dx .$$

example (of what could go wrong) rather the proof:

Let $f_j(x) = j e^{-jx}$, then

$$\int_0^{2\pi} f_j(x)\,dx = \int_0^{2\pi} j e^{-jx}\,dx = (1 - e^{-2\pi j}) \to 1$$

However $\lim_{j\to\infty} f_j(x) = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0. \end{cases}$

Hence Fatou is an inequality in this case.

Theorem 13 (i) (Monotone convergence) If $0 \le f_1(x) \le f_2(x) \le \cdots f_j(x) \le \cdots$ are increasing, and $\lim_{j\to\infty} f_j(x) = f(x)$ then

$$\lim_{j\to\infty} \int_0^{2\pi} f_j(x)\,dx = \int_0^{2\pi} \lim_{j\to\infty} f_j(x)\,dx = \int_0^{2\pi} f(x)\,dx.$$

(ii) (Dominated convergence) Suppose that $|f_j(x)| \le g(x), \forall j$ and that $\int_0^{2\pi} g(x)\,dx < +\infty$. If $\lim_{j\to\infty} f_j(x) = f(x)$ then

$$\lim_{j\to\infty} \int_0^{2\pi} f_j(x)\,dx := \int_0^{2\pi} \lim_{j\to\infty} (f_j(x))\,dx = \int_0^{2\pi} f(x)\,dx.$$

In case $g(x) = C$ a constant, the result is called the bounded convergence theorem.

Math 4FT
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(iii) $L^2(\mathbb{T}^1)$, the Lebesgue integral, and completeness

The space $L^2(\mathbb{T}^1) = \{ f(x) \text{ measurable}; \ \|f\|^2 = \int_0^{2\pi} |f(x)|^2 dx < +\infty \}$

Proposition 1 $L^2(\mathbb{T}^1)$ is a (pre)Hilbert space, in that it has a scalar product, and is closed under addition and scalar multiplication.

proof: Suppose $f \in L^2$, then $\int_0^{2\pi} |\alpha f(x)|^2 dx = |\alpha|^2 \int_0^{2\pi} |f(x)|^2 dx < +\infty$,
and
$$\int_0^{2\pi} |f+g|^2 dx \le \int_0^{2\pi} 2|f|^2 + 2|g|^2 dx \le 2(\|f\|^2 + \|g\|^2) < +\infty$$

The scalar product $(f,g) := \int_0^{2\pi} f(x) \overline{g(x)} dx$ is well defined.
- Pointwise in x, $|2 f(x) \overline{g(x)}| \le 2|f(x)||g(x)| \le |f(x)|^2 + |g(x)|^2$
therefore
$$\left| \int_0^{2\pi} f(x) \overline{g(x)} dx \right| \le \int |f(x) \overline{g(x)}| dx \le \int_0^{2\pi} |f(x)|^2 dx + \int_0^{2\pi} |g(x)|^2 dx$$
- The algebraic properties of the scalar product are also easy:
$$(f,g) = \int_0^{2\pi} f(x) \overline{g(x)} dx = \overline{\int_0^{2\pi} g(x) \overline{f(x)} dx} = \overline{(g,f)},$$
etc. $\square$

Since it is a well defined scalar product, the Schwarz inequality holds

Proposition 2 (i) $|(f,g)| = \left| \int_0^{2\pi} f(x) \overline{g(x)} dx \right| \le \left(\int_0^{2\pi} |f(x)|^2 dx \right)^{1/2} \left(\int_0^{2\pi} |g(x)|^2 dx \right)^{1/2}$

proof: The Schwarz inequality.

(ii) triangle inequality (Minkowski inequality) $\|f+g\| \le \|f\| + \|g\|$

Checking properties:
- $(f,g) = \overline{(g,f)}$
- $(\alpha f + \beta g, h) = (\alpha f, h) + \beta (g,h) = (f, \bar{\alpha} h) + (g, \bar{\beta} h)$
- $(f,f) = \|f\|^2 \ge 0$

and if $(f,f) = 0$ then $f = 0$

Note: In L^2, if $f(x) = 0$ a.e. (that is, $\text{meas}\{x : f(x) \ne 0\} = 0$) then $f = 0$ (identified with zero).

With our scalar product we define the angle between two functions $f, g \in L^2$

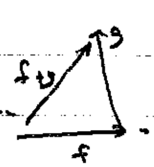
$$\mathrm{re}\,(f,g) = \|f\|\,\|g\|\cos(\theta).$$

Using this definition

$$\|f+g\|^2 = \|f\|^2 + 2\,\mathrm{re}(f,g) + \|g\|^2$$
$$= \|f\|^2 + 2\|f\|\,\|g\|\cos(\theta) + \|g\|^2 ;$$

therefore if $\cos(\theta) = 0$, $\theta = \pi/2$, then

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2$$ Pythagorean theorem.



Convergence of sequences of functions: in the topology of $L^2(\mathbb{T}^1)$, we say $f_n(x) \xrightarrow{L^2} f(x)$ when

$$\lim_{n\to\infty} \|f_n(x) - f(x)\| = \lim_{n\to\infty} \left(\int_0^{2\pi} |f_n(x) - f(x)|^2 dx \right)^{1/2} = 0.$$

Proposition 3 (other types of convergence, and their comparison)

(i) If $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ uniformly, then also

$$\lim_{n\to\infty} \left(\int_0^{2\pi} |f_n(x) - f(x)|^2 dx \right)^{1/2} = 0, \quad L^2 \text{ convergence}$$

proof Prob #2 of homework.

Q. If $f_n(x) \to f(x)$ uniformly, then does $f_n(x) \to f(x)$ pointwise?
If $f_n(x) \to f(x)$ pointwise, then does $f_n(x) \to f(x)$ uniformly?
- does $f_n(x) \to f(x)$ in L^2?

also Prob #2 of homework.

ex: A sequence which converges in L^2, but does not converge pointwise.
- let $h_n(x) = \chi_{[a_n, a_n + \frac{1}{n}]}(x)$ $0 \le x < 2\pi$, continued periodically.
- Now choose a_n dense on $[0, 2\pi)$ (and in a way in which each x is covered by some interval $[a_n, a_n + \frac{1}{n}]$ infinitely often.

Then $h_n(x) \to 0$ (since $\left(\int_0^{2\pi} |h_n(x)|^2 dx\right)^{1/2} = \frac{1}{\sqrt{n}} \to 0$) but $h_n(x)$ no limit $\forall x$.

Theorem 4 The space $L^2(\mathbb{T}^1)$ is complete.

That is, we must show that every Cauchy sequence $\{f_n(x)\}_{n=1}^{\infty}$ in L^2 has a limit $f(x) \in L^2$.

proof: Step 1 - , it suffices to consider a subsequence ~~it suffices to consider a subsequence~~. Let $\varepsilon = \frac{1}{2^j}$:

Choose N_1 so that $\forall n > N_1$

$$\|f_{N_1} - f_n\|^2 \le \frac{1}{2}$$

Now choose $N_2 > N_1$ so that $\forall n > N_2$,

$$\|f_{N_2} - f_n\|^2 \le 1/4$$

Recursively choose $N_j > N_{j-1}$ so that for all $n > N_j$,

$$\|f_{N_j} - f_n\| \le 1/2^j.$$

The subsequence $\{f_{N_j}(x)\}_{j=1}^{\infty}$ satisfies $\|f_{N_j} - f_{N_k}\|^2 \le 2^{-j}$ where $k \ge j$.

Step 2 - find a candidate limit function:

- Define the bad sets $A_j = \{x : |f_{N_{j+1}}(x) - f_{N_j}(x)| > 2^{j/3}\}$, sets where the $f_{N_j}(x)$ are not converging fast.

- Define $\mathcal{B} = \bigcap_{j \ge 1} \left(\bigcup_{k \ge j} A_k \right) := A_j$ "i.o.", the bad set, on which $f_{N_j}(x)$ are not converging.

If $x \in [0, 2\pi) \setminus \mathcal{B}$, then for some possibly very large $j > 0$, $x \notin A_k$ for all $k \ge j$. Hence

$$|f_{N_{k+1}}(x) - f_{N_k}(x)| < 2^{-k/3} \qquad \forall k \ge j.$$

- Use the Chebyshev inequality, finding

$$m(A_j) \le (2^{j/3})^2 \|f_{N_{j+1}} - f_{N_j}\|^2 \le 2^{-j/3},$$

therefore $\sum_{j=1}^{\infty} m(A_j) < 2^{4/3}$ · finite.

Hence by the Borel - Cantelli lemma, $m(B) = 0$. Therefore the set C on which $f_{N_j}(x)$ fails to converge to some limit value "$f(x)$" satisfies $m(C) = 0$ (since $C \subseteq B$).

Define $f(x) = \begin{cases} \lim_{j\to\infty} f_{N_j}(x) & x \notin C \\ 0 & x \in C \end{cases}$, our candidate limit function.

That is, $f_{N_j}(x) \to f(x)$ pointwise a.e.,
certainly for all $x \in [0, 2\pi) \setminus C$.

Step 3 By Fatou's lemma,

$$\| f_n - f \|^2 = \int_0^{2\pi} \lim_{j\to\infty} |f_n(x) - f_{N_j}(x)|^2 \, dx$$

$$\le \varliminf_{j\to\infty} \int_0^{2\pi} |f_n(x) - f_{N_j}(x)|^2 \, dx$$

If $n \geq N_\ell$, then this is $\le 2^{-\ell}$, because the sequence is Cauchy and the N_j were chosen specially.

Therefore $\|f\| \le \|f_{N_1}\| + \|f_{N_1} - f\| \le \|f_{N_1}\| + \frac{1}{\sqrt{2}} < +\infty$, so $f \in L^2$ and $f_n \xrightarrow{L^2} f$. $\boxtimes$