Math 4FT: Problem Set 3.

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Due Monday April 8, 2013.

Problem 1. Show that the space $L^1(\mathbb{T}^1)$ is not a Hilbert space.
Hint: Any two elements in a Hilbert space satisfy the parallelogram law. Namely, there is a relation between the length of adjacent sides of any parallelogram and its two diagonals;

$$\|f\|^2 + \|g\|^2 = \frac{1}{2}\Big(\|f+g\|^2 + \|f-g\|^2\Big) \ .$$

Show that there are $f, g \in L^1$ such that this relation between L^1-norms does not hold.

Problem 2. Find a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that the limit $(\lim_{n\to\infty} s_n)$ does not exist, but the limit of arithmetic means do exist

$$\lim_{j\to\infty}\Big(\frac{1}{n}\sum_{j=0}^{n-1} s_j\Big) = p \ .$$

This problem has to do with the differences between regular summability of series and Cesaro summability.

Problem 3. (i) Derive the form of the Poisson kernel $P(r,\vartheta)$ for the disk $D_1 = \{x^2+y^2 \le 1\}$ as given in class.
(ii) Complete the proof of the fact that the harmonic extension $u(x,y)$ of $f \in C(\mathbb{T}^1)$ to D_1 takes on the correct boundary data, namely in polar coordinates

$$\lim_{r\to 1}\int_{-\pi}^{+\pi} P(r,\varphi-\vartheta)f(\varphi)d\varphi = f(\vartheta) \ .$$

Problem 4. (i) Given the planar lattice

$$\Gamma := \{j(1,0) + \ell(a,b) \ : \ j,\ell \in \mathbb{Z}\}$$

with the two generators $(1,0)$ and (a,b) (with $b \neq 0$), find the dual lattice Γ'.
(ii) Give an expression for the Fourier series coefficients of a function f which is defined and periodic on the torus $\mathbb{T}^2_{ab} := \mathbb{R}^2/\Gamma$. State the Plancherel identity in this case.

Problem 5*. Let γ be a simple closed curve in $\mathbb{R}^2$ parametrized by $-\pi \le s \le +\pi$; $\gamma(s) = (x(s), y(s))$. Define ℓ to be the length of γ, and let

$$d := \sup_{-\pi\le s_1,s_2\le+\pi} |\gamma(s_1) - \gamma(s_2)|$$

be the diameter.
(i) Show that the Fourier coefficients satisfy

$$\big(|\hat{x}_k|^2 + |\hat{y}_k|^2\big)^{1/2} \le \frac{d}{\sqrt{8\pi}} \ .$$

(ii) If γ encloses a convex region, show that

$$2d \le \ell \le \pi d \ .$$