Math 4FT: Problem Set 1.

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Problem 1. Prove the convolution theorem for Fourier series. Namely, show that if two 2π-periodic functions $f(x)$ and $g(x)$ are $C^2(\mathbb{S}^1)$ with Fourier coefficients $\{\hat{f}_k\}_{-\infty<k<+\infty}$ and $\{\hat{g}_k\}_{-\infty<k<+\infty}$ respectively, then the product $fg(x)$ is also $C^2(\mathbb{S}^1)$ and 2π-periodic, with Fourier coefficients given by the convolution product

$$\widehat{(fg)}_k = \frac{1}{\sqrt{2\pi}} \sum_{-\infty<\ell<+\infty} \hat{f}_{k-\ell}\hat{g}_\ell := \hat{f} * \hat{g}_k \ .$$

Show that the convolution product $\hat{f} * \hat{g}$ is commutative and associative.

Problem 2. Derive an expression for the heat kernel with Neumann boundary conditions on the interval $x \in [0, \pi]$. That is, the kernel of the solution operator for heat flow of the following problem:

$$\begin{aligned} &\partial_t u = \frac{1}{2}\partial_x^2 u \\ &-\partial_x u(0,t) = 0 = \partial_x u(\pi, t) \\ &u(x,0) = f(x) \ . \end{aligned}$$

That is, find the kernel $h_N(x, y, t)$ such that the solution of this problem is given by

$$u(x,t) = \int_0^\pi f(y) h_N(x,y,t)\, dy \ .$$

Show that for $t > 0$, for all $x, y \in [0, \pi]$ then $h(x - y, t) < h_N(x, y, t)$, and thus the maximum principle holds.

Problem 3. Use the heat kernel h_D to prove an identity similar to the Jacobi identity for the ϑ-function;

$$\frac{2}{\pi}\sum_{k=1}^{+\infty} e^{-\frac{k^2 t}{2}} \sin(kx)\sin(ky) = \frac{1}{\sqrt{2\pi t}} \sum_{-\infty m<+\infty} e^{-(x-y-2\pi m)^2/2t} - e^{-(x+y-2\pi m)^2/2t} \ .$$

What is the analog identity for the Neumann heat kernel $h_N(x, y, t)$?

Problem 4. The time-periodically forced heat equation.
Let $u(x, t)$ be the temperature in a body represented as $0 < x < +\infty$ (modeling, for example, the earth, for which x represents depth) which is being heated, 2π periodically in time, on the boundary $x = 0$ (for example, by the seasonal variations of heat flux from the sun. That is, suppose that the solution u to the heat equation

$$\partial_t u = \frac{1}{2}\partial_x^2 u \ , \quad u(0,t) = f(t) = f(t + 2\pi) \ ,$$

satisfies $u(x, t) = u(x, t+2\pi)$ and furthermore $u(x, t) \to 0$ as $x \to +\infty$. Find the solution by Fourier series, writing

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty<k<+\infty} c_k(x) e^{ikt} \ ,$$

and solving for $c_k(x)$. Describe the damping factor and the phase shift of each Fourier coefficient $c_k(x)$. If $f(t) = \sin(t)$, what are the depths at which the phase shift is precisely π, namely depths at which the temperature is maximally cooler in summer and warmer in winter.