Math 4FT: Final exam.

April 8, 2013
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Due Friday April 26

Problem 1. Show that the convergence properties of the Fourier transform are local, despite the fact that the Fourier transform of a function $f(x)$ takes into account global information about f. Namely, show that if $f(x)$ is differentable at a point $x_0 \in \mathbb{T}^1$, then the partial sums $S_n(f)(x_0)$ converge to $f(x_0)$ as $n \to +\infty$. In particular this shows that the function $g(x)$ that was constructed in class as a lacunary series is not differentiable at a dense set of points.

Problem 2. The set of continuous periodic functions $C(\mathbb{T}^1)$ is a subset of the space of bounded (and Lebesgue measurable) periodic functions $L^\infty(\mathbb{T}^1)$. They share the topology of uniform convergence, expressed by the norm

$$\|f\|_\infty = (\mathrm{ess})\sup_{x\in\mathbb{T}^1}|f(x)| \ .$$

(i) Is translation continuous on $C(\mathbb{T}^1)$? That is, is it true for all $f \in C(\mathbb{T}^1)$ that

$$\lim_{y\to 0} \|f(\cdot - y) - f(\cdot)\|_\infty = 0 \ .$$

(i) Is translation continuous on $L^\infty(\mathbb{T}^1)$? That is, is it true for all $f \in L^\infty(\mathbb{T}^1)$ that

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Problem 3. The Sobolev space $H^s(\mathbb{T}^1)$ consists of those periodic functions $f(x)$ such that $f \in L^2$, $\partial_x f \in L^2$, $\ldots \partial_x^s f \in L^2$. The Sobolev norm of f is given by the expression

$$\|f\|_s^2 := \int_{\mathbb{T}^1} |f(x)|^2 + |\partial_x^s f(x)|^2 \, dx \ .$$

(i) Use the Plancherel identity to show that

$$\|f\|_s^2 = \sum_{k\in\mathbb{Z}^1} (1 + |k|^{2s})|\hat{f}_k|^2 \ .$$

(ii) Prove the Sobolev inequality, that for all $x \in \mathbb{T}^1$

$$|f(x)| \le C_s \|f\|_s \ ,$$

for any $s \ge 1$. The conclusion is that $L^\infty(\mathbb{T}^1) \subseteq H^1(\mathbb{T}^1)$ for $s \ge 1$. Is $L^\infty \subseteq L^2(\mathbb{T}^1)$?

Problem 4. The space of integrable functions $L^1(\mathbb{T}^1)$ is an algebra under the operations of addition and convolution product

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x-y)g(y)\, dy \ .$$

(i) Show that

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1 \ .$$

(ii) Show that $L^2 \subseteq L^1$ is an *ideal* of L^1, meaning that $f * g \in L^2$ as long as one of the two factors f or $g \in L^2$.

(iii) Show that L^1 does not have a multiplicative identity element, namely that there is no function $e(x) \in L^1$ such that for all $f \in L^1$ then

$$f * e(x) = f(x) \ .$$