C*-algebras - a case study

Definition

Suppose that H is a Hilbert space. A C*-algebra is an operator-norm closed $*$-subalgebra of $B(H)$.

- C*-algebras are closed under ultraproducts and subalgebras so they should be captured by continous model theory.
- We introduce operator-norm balls of fixed radius as sorts and piece the algebra together. We have function symbols for $+$, $\times$ and scalar multiplication as well as * . The metric comes from the operator norm.
- Here is a list of axioms:

Axioms for C*-algebras

- $x + (y + z) = (x + y) + z$, $x + 0 = x$, $x + (-x) = 0$ (where $-x$ is the scalar -1 acting on x), $x + y = y + x$, $\lambda(\mu x) = (\lambda\mu)x$, $\lambda(x + y) = \lambda x + \lambda y$, $(\lambda + \mu)x = \lambda x + \mu x$.
- $1x = x$, $x(yz) = (xy)z$, $\lambda(xy) = (\lambda x)y = x(\lambda y)$, $x(y + z) = xy + xz$;
- $(x^*)^* = x$, $(x + y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$
- $d(x, y) = d(x - y, 0)$; we write $\|x\|$ for $d(x, 0)$.
- $\|xy\| \leqslant \|x\|\|y\|$
- $\|\lambda x\| = |\lambda|\|x\|$
- $\|x^* x\| = \|x\|^2$ (C*-indentity)
- $\sup_{a \in B_1} \|a\| \leqslant 1$

Two important theorems

Theorem (Spectral Theorem)

Suppose that A is an abstract C-algebra and $a \in A$ is self-adjoint. Then the abstract C*-subalgebra $C^*(a)$ generated by a and the identity on A is isomorphic to $C(sp(a))$ via an isomorphism sending a to $id_{sp(a)}$ and id_A to the constant function 1.*

Theorem (Gel'fand-Naimark)

Any abstract C-algebra A is *-isomorphic to a C*-algebra of operators on a Hilbert space.*

Sketch of a proof of GN

Definition

Suppose that A is a C*-algebra. A state on A is a linear functional $\lambda : A \to C$ such that

1. $\lambda(1) = 1$ and
2. $\lambda(a^* a) \geqslant 0$ for all $a \in A$ (λ is positive).

Lemma

For any C-algebra A and any positive $a \in A$, there is a state on A such that $\lambda(a) > 0$.*

- Suppose that λ is a state on a C*-algebra A. We define $\langle a, b \rangle = \lambda(b^* a)$ for $a, b \in A$.
- Claim: $\langle \cdot, \cdot \rangle$ is a pseudo-inner product i.e. it is an inner product except for the fact that $\langle a, a \rangle = 0$ may not imply $a = 0$.

Sketch of a proof of GN

- Let $I = \{a \in A : \langle a, a\rangle\}$. Claim: I is a left ideal of A.
- Consider A/I which is an inner product space on which A acts and so A acts on the completion of A/I which is a Hilbert space.
- To prove the GN-theorem, consider all states on A and create H, the direct sum of all the associated Hilbert spaces that we just constructed.
- A acts faithfully on H since if $ax = 0$ for all $x \in H$, we have that $\lambda(a^* a) = 0$ for all states λ.
- But then $a^* a = 0$ and by the C*-identity, $\|a\| = 0$ which implies that $a = 0$.

Choosing the right language

- Remember that we guess that C*-algebras should be universally axiomatized - the class of C*-algebras was closed under ultraproducts and subalgebras. There is a subtle problem with this; it has to do with the choices of ranges for our function symbols.
- Suppose we are considering a normed linear space and we wish to assert that the unit ball is convex.
- $+$ when restricted to the unit ball would most naturally map to the ball of radius 2; scalar multiplication by 1/2 could map the ball of radius 2 into the unit ball and so $(x + y)/2$ sends the unit ball to itself.
- The syntax guarantees that the unit ball is convex!

Choosing the right language, cont'd

- If on the other hand, the scalar 1/2 on B_2 had range B_2 then $(x+y)/2$ syntactically map the unit ball to B_2.
- We would need to have an axiom that said this term has range in the unit ball

$$\sup_{x,y \in B_1} \inf_{z \in B_1} (x+y)/2 = i(z)$$

where i is the inclusion map from B_1 to B_2; this axiom is not universal.

The new language for C*-algebras

- For every *-polynomial $p(x)$ we introduce a term $\tau(x)$.
- We must specify the range of τ on inputs from balls B_i. Let $\mathcal{C}$ be the class of C*-algebras.
- The range then is B_N where N is

$$\min\{n : \text{for all } A \in \mathcal{C}, a \in B_i(A), \|p(a)\| \leqslant n\}$$

- We introduce universal axioms that say that if B_j is the range of p and i is the inclusion map from B_N to B_j then for all $x \in B_i$, $i(\tau(x)) = p(x)$.
- T_{C*} will be all the former axioms and all these new ones as well in this expanded language.

Models of T_{C*}

- If A is a C*-algebra it is clear that there is a model $M(A)$ associated to it which satisfies T_{C*} - the sorts B_n are interpreted as the balls of operator norm $\leqslant n$ and all the functions are interpreted naturally.
- Moreover, if M is a model of T_{C*}, by the Gel'fand-Naimark theorem, we can reconstruct a C*-algebra A_M by piecing together the sorts.
- To see that this is an equivalence of categories, we need to show that $M(A_M)$ is isomorphic to M.
- The only real question is why the operator norm unit ball A_M is exactly the sort B_1 in M - the axioms guarantee that $B_1(M)$ is contained in the unit ball; we need to see that they are equal.

A little calculation

- Suppose that $u \in M$, $\|u\| \leqslant 1$ and $u \in B_n(M)$.
- Let
$$t_n(x) = \begin{cases} 1 & 0 \leqslant x \leqslant 1 \\ \frac{1}{\sqrt{x}} & 1 < x \leqslant n \end{cases}$$
and consider $f(u) = ut_n(u^*u)$.
- To compute the norm of $f(u)$ for $\|u\| \leqslant n$, we see that $\|f(u)\|^2 = \|t_n(u^*u)u^*ut_n(u^*u)\| = \|g(u^*u)\|$ where $g(x) = xt_n^2(x)$.
- Since
$$g(x) = \begin{cases} x & 0 \leqslant x \leqslant 1 \\ 1 & 1 < x \leqslant n \end{cases}$$
we obtain that the norm of $f(u)$ is at most 1 when $\|u\| \leqslant n$.

A little calculation, cont'd

- Now fix polynomials $p_k(x)$ which tend to $t_n(x)$ from below on the interval $[0, n]$.
- By doing a calculation similar to the one on the previous slide, the *-polynomial $up_k(u^*u)$ sends operators of norm $\leqslant n$ to operators of norm $\leqslant 1$.
- Now because of our choice of language, we have a term which equals $xp_k(x)$ and which sends B_n to B_1 in all models of T_{C*}.
- Since $up_k(u)$ tends to u as k tends to infinity, we conclude that in fact, $u \in B_1(M)$.

Some topologies on $B(H)$

- We get on topology on $B(H)$ arising from the operator norm - this is called the norm, sup or uniform topology.
- Pointwise convergence of operators on $B(H)$ is called the strong operator topology. A basic open set has the following form: fix $a \in B(H)$, $x \in H$ and $\epsilon > 0$ and let

$$U = \{b \in B(H) : \|ax - bx\| < \epsilon\}$$

- A third topology is the weak operator topology; its basic open sets are given by: fix $a \in B(H)$, $x, y \in H$ and $\epsilon > 0$ and let

$$U = \{b \in B(H) : |\langle ax, y\rangle - \langle bx, y\rangle| < \epsilon\}$$

- The norm topology contains the strong topology which contains the weak topology.
- The operator norm unit ball of $B(H)$ is compact in the weak topology.

Von Neumann algebras

Definition

A von Neumann algebra $A \subseteq B(H)$ is a *-subalgebra of $B(H)$ containing 1 which is closed in the weak operator topology.

- $M_n(C)$ is a von Neumann algebra as is $B(H)$ for any Hilbert space H.
- $L^\infty([0,1])$ of bounded, integrable functions acting on $L^2([0,1])$.
- A non-commutative infinite-dimensional example is R which can be thought of as the weak closure of

$$M_2(C) \hookrightarrow M_4(C) \hookrightarrow \ldots$$

or alternatively, one can think of this as $M_2(C) \otimes M_2(C) \otimes \ldots$ acting on $C^2 \otimes C^2 \otimes \ldots$.

Two important theorems

Definition

If $A \subseteq B(H)$ then $A' = \{b \in B(H) : ab = ba \text{ for all } a \in A\}$. We call A' the commutant of A.

Theorem (Double commutant theorem)

*The following are equivalent for a *-subalgebra of $B(H)$:*

1. *A is a von Neumann algebra.*
2. *A is closed in the strong operator topology.*
3. *$A'' = A$.*

Theorem (Kaplansky density theorem)

*Suppose that $A \subseteq B(H)$ is a *-subalgebra and M is the weak closure of A. Then the norm unit ball of A is strongly dense in the norm unit ball of M.*

Tracial von Neumann algebras

Definition

We say that a von Neumann algebra A is tracial if there is a weakly continuous state $tr : A \to C$ which is faithful ($tr(a^*a) = 0$ implies $a = 0$) and satisfies $tr(aa^*) = tr(a^*a)$.

- $M_n(C)$ is a tracial von Neumann algebra - normalize the usual trace by dividing by n.
- For infinite dimensional H, $B(H)$ is not tracial.
- $L^\infty([0,1])$ of bounded, measurable functions acting on $L^2([0,1])$ is tracial - $tr(f) = \int_0^1 f\,dx$.
- R is tracial - the trace is the limit of the traces on M_{2^n}.
- Is the class of tracial von Neumann algebras elementary in continuous logic?

The case for tracial von Neumann algebras

- The class of tracial von Neumann algebras is going to be closed under any reasonable notion of subalgebra.
- What can we say about ultraproducts? There does exist a notion of tracial ultraproduct:
- If U is an ultrafilter on I and M_i are tracial von Neumann algebras with traces τ_i then one constructs the usual operator algebra ultraproduct of the M_i's and defines $tr(\bar{a}) = \lim_{i \to U} \tau_i(a_i)$.
- One checks that this is a trace and makes the ultraproduct a tracial von Neumann algebra.
- So we have a class closed under subalgebras and ultraproducts; how do we capture it in continuous logic?

The language for tracial von Neumann algebras

- The language for tracial von Neumann algebras will include the language we used for C*-algebras.
- There will be sorts B_n for $n \in N$ and sorted function symbols for $+, \times, *$ and scalar multiplication. We will also have constants 0 and 1.
- We will have relation symbols for the trace. Like with the Hilbert space example, we will need to have this sorted and we will need to consider the real and imaginary parts.
- In order to get a universal axiomatization, we will need to introduce ”optimized” terms which represent the *-polynomials as we did in the C*-algebra case.

The standard model

- If A is a tracial von Neumann algebra, we create a metric structure in the language of tracial von Neumann algebras as follows:
- B_n is interpreted as the *operator norm* ball of radius n; notice that the operator norm is not part of the language - some words are in order ...
- The metric is given on each ball by the metric induced by the trace.
- All of the algebraic functions (including the optimized terms) are interpreted in the usual manner.
- The trace supplies an interpretation for the real and imaginary parts of the trace relation.
- We will call this metric structure $M(A)$.

Axioms for tracial von Neumann algebras

- The have axioms that make our models complex *-algebras.
- There are axioms expressing the properties of the trace function: trace is linear, $tr(a^*a) \geqslant 0$ and $tr(aa^*) = tr(a^*a)$.
- Trace is related to the metric; let $\|a\|_2 = d(a, 0)$, we have $d(a, b) = \|a - b\|_2$ and $\|a\|_2^2 = tr(a^*a)$.
- For every $n, m \in N$,

$$\sup_{a \in B_n} \sup_{x \in B_m} \max\{0, \|ax\|_2 - n\|x\|_2\}$$

- Finally we add axioms relating *-polynomials to their associated optimized term.

Correctness of the axioms

- If A is a tracial von Neumann algebra then it is straightforward to check that $M(A)$ is a model of the axioms.
- If M is a model of the axioms then putting the sorts together we can create a *-algebra A_M.
- The only question is why $M(A_M)$ is isomorphic to A.
- The fact that the unit ball is correctly interpreted is handled exactly like in the case of C*-algebras - we have all the necessary terms.
- The Kaplansky density theorem tells us that to see that A_M is a von Neumann algebra we only need to see that the unit ball is closed in the strong topology.
- But B_1 is complete with respect to the metric which is given by the trace so a quick calculation shows that B_1 is closed in the strong topology and we are done.