From last time

Theorem

The following are equivalent:

1. *T is stable.*
2. *T does not have the order property.*
3. *T supports a stationary independence relation.*
4. *(L separable) For all (any) separable models of T, the ultrapowers with respect to non-principal ultrafilters on $\mathbb{N}$ are necessarily isomorphic.*

From last time, cont'd

Definition

Suppose that $(I, <)$ is a linear order and $\langle \bar{a}_i : i \in I \rangle$ is an I-indexed sequence in some model M. Then this sequence is said to be indiscernible if whenever $i_1 < i_2 < \ldots < i_n$ and $j_1 < j_2 < \ldots < j_n$ then $t(a_{i_1} \ldots a_{i_n}) = t(a_{j_1} \ldots a_{j_n})$.

Theorem

Suppose that M is a non-compact metric structure. Then for any $(I, <)$ there is an $M' \models Th(M)$ and an I-indexed non-constant indiscernible sequence in M'.

- Example: In an infinite dimensional Hilbert space, an orthonormal set is indiscernible ordered any way you like.
- The sequence which witnessed the order property for Urysohn space was also indiscernible ordered in the way it was given.

Order implies unstable

Corollary (to the previous proof)

If T has the order property then T is unstable.

- Proof sketch: Fix $\varphi(\bar{x}, \bar{y})$ and $r < s$ which witnesses the order property. Using the same style of proof from the previous theorem we can prove that for any ordered set $(I, <)$, we can find $M \models T$ and I-indexed indiscernible sequence $\langle a_i b_i : i \in I \rangle$ such that $\varphi(a_i, b_j) \leqslant r$ if $i \leqslant j$ and $\varphi(a_i, b_j) \geqslant s$ if $i > j$.
- Now fix a cardinal λ and choose κ least such that $2^\kappa > \lambda$. Then $\kappa \leqslant \lambda$ and $2^{<\kappa} \leqslant \lambda$.
- Order 2^κ by $\eta < \mu$ if, for the greatest α such that $\eta\restriction_\alpha = \mu\restriction_\alpha$, $\eta(\alpha) < \mu(\alpha)$.
- Identify $2^{<\kappa}$ with those elements of 2^κ which are eventually 0.

Proof cont'd

- Pick a model M and a 2^κ-indexed indiscernible sequence $\langle a_\eta b_\eta : \eta \in 2^\kappa \rangle$ ordered by φ.
- Let $B = \{b_\eta : \eta \in 2^{<\kappa}\}$, a set of size $\leqslant \lambda$.
- For $\eta \in 2^\kappa \backslash 2^{<\kappa}$, consider all the types $t(a_\eta/B)$. Now if $\eta < \mu$, choose $\bar{\mu} \in 2^{<\kappa}$ such that $\eta < \bar{\mu} < \mu$. Then

$$\varphi(a_\eta, b_{\bar{\mu}}) \leqslant r \text{ and } \varphi(a_\mu, b_{\bar{\mu}}) \geqslant s$$

- So $t(a_\eta/B)$ and $t(a_\mu/B)$ are not equal. Moreover, if $\epsilon = \frac{s-r}{2}$ and we choose δ from the continuity modulus for φ in the $\bar{x}$-variable, we see that $t(a_\eta/B)$ and $t(a_\mu/B)$ are at least δ apart so $\chi(S(B)) \geqslant 2^\kappa > \lambda$ and $\chi(B) \leqslant \lambda$.
- B isn't a model but we could extend B to a model of the same density character and we would still have too many separated types over this model.
- So the order property implies that T is not λ-stable for any λ.

Corollary

- In fact, we can say more (and this will be useful later): The previous proof showed that if φ has the order property then for every λ there is a model M_λ with $\chi(M_\lambda) \leqslant \lambda$ such that $\chi(S_\varphi(M_\lambda)) > \lambda$ where $S_\varphi(M)$ is the set of φ-types over M. The metric on this space of types is given by

$$\sup_{b \in M} \left| p^{\varphi(x,b)} - q^{\varphi(x,b)} \right|$$

Unstable implies order

Definition

We say $p(x) \in S(M)$ is finitely determined if for every formula $\varphi(x, y)$ and every $\epsilon > 0$ there is a finite $B \subseteq M$ and $\delta > 0$ such that for all $c_1, c_2 \in M$, if

$$\max_{b \in B} |\varphi(b, c_1) - \varphi(b, c_2)| < \delta$$

then

$$|p^{\varphi(x, c_1)} - p^{\varphi(x, c_2)}| \leqslant \epsilon$$

Theorem

The following are equivalent:

1. *T is stable.*
2. *T does not have the order property.*
3. *For every $M \models T$, every type in $S(M)$ is finitely determined.*

Unstable implies order: proof

- We just proved that 1 implies 2. Let's show that 3 implies 1.
- Fix λ such that $\lambda^{\chi(L)} = \lambda$. Then if $M \models T$ and $\chi(M) \leqslant \lambda$, there are at most $\lambda^{\chi(L)} = \lambda$ many types in $S(M)$ by finite determinancy. So T is λ-stable.
- We show now that the failure of 3 implies the existence of order. So fix a type $p(x) \in S(M)$ which is not finitely determined say witnessed by a formula $\varphi(x, y)$ and $\epsilon > 0$.
- We use p to construct a sequence $a_i b_i c_i$ in M inductively; assume we have constructed these for all $i < j$.
- By assumption, we know that we can find b_j and c_j so that

$$\max_{i<j} |\varphi(a_i, b_j) - \varphi(a_i, c_j)| < \frac{\epsilon}{6} \text{ and } |p^{\varphi(x,b_j)} - p^{\varphi(x,c_j)}| > \epsilon$$

Unstable implies order: proof, cont'd

- Now by the approximate finite satisfiability of p, we can find $a_j \in M$ so that for all $i \leqslant j$

$$|\varphi(a_j, b_i) - p^{\varphi(x,b_i)}| \leqslant \frac{\epsilon}{3} \text{ and } |\varphi(a_j, c_i) - p^{\varphi(x,c_i)}| \leqslant \frac{\epsilon}{3}$$

- So we have that if $i < j$

$$|\varphi(a_i, b_j) - \varphi(a_i, c_j)| \leqslant \frac{\epsilon}{6}$$

and for $i \geqslant j$,

$$|\varphi(a_i, b_j) - \varphi(a_i, c_j)| \geqslant \frac{\epsilon}{3}$$

- If we let $\theta(x_1 y_1 z_1, x_2 y_2 z_2) := |\varphi(x_1, y_2) - \varphi(x_1, z_2)|$ then θ orders the sequence $\langle a_i b_i c_i : i \in \mathbb{N} \rangle$.

Dividing

- We now focus on showing that a stable theory has a stationary independence relation. We fix a saturated model M and work entirely inside M.
- If $p \in S(A)$ and $A' = \sigma(A)$ for some automorphism of M σ, we write $p_{A'}$ for $\sigma(p)$.

Definition

We say that a satisfiable partial type $p(x) \in S(B)$ divides over A if there is an A-indiscernible sequence $\langle B_i : i \in \mathbb{N} \rangle$ such that $B_0 = B$ and $\bigcup \{p_{B_i} : i \in \mathbb{N}\}$ is not satisfiable.

Examples of dividing/non-dividing

- In discrete model theory, we can consider an algebraically closed field and a type $p(x)$ over A. p does not divide over the empty set iff p says x is algebraic over A only when x was already algebraic over the empty set.
- More generally if above, $p(\bar{x})$ is a type in more than one variable over A then p does not divide over the empty set iff p implies that the transcendence degree of $\bar{x}$ over A is the same as the transcendence degree of $\bar{x}$ over the empty set.
- In the theory of infinite dimensional Hilbert spaces, a type $p(x)$ over a parameter set A does not divide over the empty set iff any realization of p is orthogonal to A.
- In the theory of Urysohn space, $p(x)$ over a single point a does or does not divide over the empty set depending on $d(x, a)$.

Dividing as independence

- For any theory, we will define $A \downarrow_C B$ to mean $t(A/BC)$ does not divide over C. What properties does this relation have?
- Invariance is immediate from the definition.
- Finite character follows from compactness.
- The direction of transitivity that says that if $C \subseteq D$ and $A \downarrow_B D$ then $A \downarrow_B C$ and $A \downarrow_{BC} D$ is also immediate.
- Local character, symmetry and stationarity will have to wait until we assume stability. Let's look at transitivity and extension. First we prove a lemma.

A lemma

Lemma

The following are equivalent:

1. $a \downarrow_A b$
2. *For every A-indiscernible sequence I such that* $b \in I$ *there is* $a' \equiv_{Ab} a$ *such that I is* Aa'*-indiscernible.*
3. *For every A-indiscernible sequence I such that* $b \in I$ *there is* $J \equiv_{Ab} I$ *such that J is Aa-indiscernible.*

- Conditions 2 and 3 are readily equivalent and 2 implies 1 is straightforward.
- We will show that 1 implies 3. For convenience, assume that A is empty and $p(x)$ is the type of a over b.
- Fix any indiscernible sequence I with $b \in I$ and we wish to find the requisite J. We do this by compactness.

A lemma, cont'd

- Write down formulas Σ that express the fact that J has the same type as I over b; J is indiscernible over a and that the type of a over c for any individual $c \in J$ is p_c.
- Now stretch I to an indiscernible sequence I' of size $\beth_\omega(2^{\chi(T)})$; this can be done by compactness.
- What we know from the fact that p does not divide over the empty set is that $\{p_c : c \in I'\}$ is satisfiable.
- By the Erdos-Rado theorem, we can show that Σ is consistent because for any n, there are at most $2^{\chi(T)}$ many n-types over a witnessed among the n-tuples from I'.

Left transitivity

Proposition

Left transitivity If $B \subseteq C \subseteq D$, $D \downarrow_C A$ and $C \downarrow_B A$ then $D \downarrow_B A$.

- Proof: we use the lemma we just proved. Fix a B-indiscernible sequence I such that $A \in J$.
- Since $C \downarrow_B A$, we can find J with the same type as I over BA and which is C-indiscernible.
- But then since $D \downarrow_C A$, we can find K with the same type as J over CA which is D-indiscernible.
- So K has the same type as I over BA and is D-indiscernible so $D \downarrow_B A$ by the lemma. $\square$
- Conclusion: If we have symmetry of dividing then transitivity is for free.

Discrete stability

Definition

We say that T is discretely λ-stable if for any M such that $\chi(M) \leqslant \lambda$, $|S(M)| \leqslant \lambda$.

Theorem

The following are equivalent:

1. *T is stable.*
2. *T is λ-stable for all λ such that $\lambda^{\chi(T)} = \lambda$.*
3. *T is discretely λ-stable for all λ such that $\lambda^{\chi(T)} = \lambda$.*

- 3 implies 2 implies 1 are straightforward.
- For 1 implies 3, remember that if T is stable then every type over a model is finitely determined.
- So, for every formula φ, $|S_\varphi(M)| \leqslant |M|^{\aleph_0}$.
- $|S(M)| \leqslant |M|^{\aleph_0 \cdot \chi(M)} = |M|$.

Local character

Theorem

If T is stable then dividing has local character.

- In fact, local character of dividing is equivalent to a notion called simplicity (more general than stability).
- We will show that if dividing fails to have local character with $\kappa = \chi(T)$ then the theory is unstable.
- Fix a type $p \in S(B)$ which divides over every subset $B_0 \subseteq B$ with $|B_0| \leqslant \kappa$.
- Start with $B_0 = \emptyset$ and define a continuous increasing chain of subsets B_α for $\alpha < \chi(T)^+$ such that
  1. $|B_\alpha| \leqslant \kappa$ and
  2. $p\restriction_{B_{\alpha+1}}$ divides over B_α for every α.

Local character, cont'd

- Choose λ such that $\lambda^{\kappa^+} > \lambda^\kappa = \lambda \geqslant 2^{\kappa^+}$.
- Now define a tree of sets $\langle B_\eta : \eta \in \lambda^{<\kappa^+} \rangle$ such that
  1. For all $\eta \in \lambda^{\kappa^+}$, $\langle B_{\eta\restriction_\alpha} : \alpha < \kappa^+ \rangle \equiv \langle B_\alpha : \alpha < \kappa^+ \rangle$ and
  2. for every $\eta \in \lambda^{<\kappa^+}$, $\langle B_{\eta \wedge i} : i < \lambda \rangle$ is indiscernible of order type λ over B_η which witnesses an automorphic copy of the fact that $p\restriction_{B_{\alpha+1}}$ divides over B_α where $\alpha = len(\eta)$.
- Let p_η for $\eta \in \lambda^{\kappa^+}$ be the automorphic image of p with domain $\bigcup_{\alpha < \kappa^+} B_{\eta\restriction_\alpha}$.
- Let P_η be a maximal mutually satisfiable collection of p_ν which contains p_η.
- $|P_\eta| \leqslant \aleph_0^{\kappa^+}$.
- It follows then that the P_η's represent, altogether, λ^{κ^+} many types over λ^κ many parameters and this contradicts discrete λ-stability.