Outline

- Some terminology
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- Two examples
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Some terminology

Definition

Suppose that $F : C \to D$ is a functor then we say that

- F is full if for all objects $c, c' \in C$, $F : Hom(c, c') \to Hom(F(c), F(c'))$ is onto.
- F is failthful if for all objects $c, c' \in C$, $F : Hom(c, c') \to Hom(F(c), F(c'))$ is injective.
- F is dense if for all objects d in D, there is $c \in C$ such that $F(c) \cong d$.
- F is an equivalence of categories if F is full, faithful and dense.

Some terminology, cont'd

- If $L \subseteq L'$ are two languages, T and T' are complete theories in L and L' respectively then we call $F : Mod(T') \to Mod(T)$ the forgetful functor if it takes $M \models T'$ to M's restriction to L.
- F is full if whenever $f : F(M) \to F(M')$ is an elementary map, it is the restriction of a map between M and M' or in other words, maps between models of T lift to maps between models of T'.
- F is faithful means that if a map between $F(M)$ and $F(M')$ lifts to one between M and M' then this lifting is unique.
- F is dense if every model of T can be expanded to a model of T'.

Conceptual completeness

- The question we are trying to understand is when two complete theories T and T' possibly in different languages L and L' have equivalent categories of models.
- We will answer the slightly simpler looking question: if $T \subseteq T'$ and $L \subseteq L'$ and the forgetful functor $F : Mod(T') \to Mod(T)$ is an equivalence of categories then what can we say about T'. The answer is that we will be able to interpret T' in T^{eq} which we still have to describe in full.
- Notice in the description of the problem that L' could have new sorts or could have new functions and relation symbols which act on the old sorts (or both of these).

Two examples: example 1

- These two examples are in the first order context to give a sense of what is possible.
- The first example should be considered in light of the fact that for continuous logic one can add countable products without changing the category of models.
- Suppose T is a complete theory in a language L and S_n for $n \in \mathbb{N}$ are countably many sorts from L.
- Let $L' = L \cup \{S, \{\pi_n : n \in \mathbb{N}\}\}$ and expand models of T, M, to an L'-structure M' by letting $S(M') = \prod_{n \in \mathbb{N}} S_n(M)$ and π_n be the projection onto the nth coordinate. Let $T' = Th(M')$.

Example 1, cont'd

- The claim is that T' does not capture the product in a first order manner or more formally that $F : Mod(T') \to Mod(T)$ is not an equivalence of categories.
- To see this, consider the set of formulas:

$$\Sigma = \{\pi_n(x) = \pi_n(y) : n \in \mathbb{N}\} \cup \{x \neq y\}$$

- Σ is consistent with T' and so there is a model of T' in which the sort S is not the product of the sorts S_n. One can check that the forgetful functor is not faithful here.

Example 2

- Suppose that T is a complete theory in a first order language L and that $E(x,y)$ is a countable partial type which defines an equivalence relation in models of T.
- For example, if $T = Th(\mathbb{R})$ and $E(x,y) = \{|x-y| \leqslant 1/n : n \in \mathbb{N}\}$ then E is such a partial type.
- The equivalence classes of a type-definable equivalence relation are called hyperimaginaries and various approaches have been taken to deal with them smoothly.
- They are important canonical objects associated to models of a first order theory. Let's see that unfortunately we can't capture them in the same manner we capture equivalence classes of definable equivalence relations.

Example 2, cont'd

- Suppose that we expand L to L' by adding a new sort S and map π.
- If M is a model of T, expand it to M' by letting $S(M')$ be the set of E-equivalence classes in M and π the canonical projection.
- Now let $T' = Th(M')$ and consider if the forgetful functor is an equivalence of categories.
- Consider the following set of formulas

$$\Sigma = \{\varphi(x, y) : \varphi \in E\} \cup \{\pi(x) \neq \pi(y)\}$$

- Σ is inconsistent with T' iff E is definable i.e. equivalent to the conjunction of finitely many formulas.
- If Σ is consistent then we see that not all models interpret S as the hyperimaginaries.
- Hyperimaginaries can be captured by continuous logic; we'll return to this.

Beth definability

- Fix a complete theory T in a language L in discrete first order logic.
- Suppose that P is a new predicate symbol and $L_P = L \cup \{P\}$.

Theorem

Beth Definability Suppose that T and T_P are complete theories in L and L_P such that $T \subseteq T_P$ and that for each $M \models T$, there is a unique way M_P to expand M to a model of T_P. Then T_P proves that P is equivalent to an L-formula.

- The theorem says in terms of forgetful functors that the forgetful functor from $Mod(T_P)$ to $Mod(T)$ is an equivalence of categories.

Proof of Beth definability

- We sketch a proof to illustrate a certain style of proof; we could also prove a version of this in continuous logic but we will prove something more general so we just highlight the proof method.
- Let

$$\Sigma = \{\varphi(\bar{x}) : T_P \models P(\bar{x}) \rightarrow \varphi(\bar{x}), \varphi(\bar{x}) \text{ is an } L\text{-formula}\}$$

- If Σ is not consistent with $\neg P(\bar{x})$ then T_P proves that P is equivalent to an L-formula; if Σ is not consistent with $P(\bar{x})$ then T_P proves P is empty. In either case, we are done and so we assume that Σ is consistent with both $P(\bar{x})$ and $\neg P(\bar{x})$.
- Let Σ^* be a maximal collection of L-formulas containing Σ and consistent with both P and $\neg P$. Such a Σ^* exists by Zorn's Lemma; let's argue that Σ^* is complete.

Proof of Beth definability, cont'd

- Now fix a saturated model M of T_P which realizes both $\Sigma^* \cup P$ and $\Sigma^* \cup \neg P$ by $\bar{a}$ and $\bar{b}$.
- Since $\bar{a}$ and $\bar{b}$ realize the same L-type there is an L-automorphism of M sending $\bar{a}$ to $\bar{b}$.
- But M restricted to L can be expanded to a model of T_P two ways: as M and as $(M{\restriction}_L, \sigma(P))$.
- They are different expansions since $\bar{b} \in \sigma(P)$ but not in P.

Stable embeddedness

- We return to continuous logic. Suppose $T \subseteq T'$, two complete theories in $L \subseteq L'$ respectively.

Definition

If M is the L-reduct of M', a model of T', we say that M is stably embedded in M' if for every formula $\varphi(\bar{x}, \bar{y})$ in L' with $\bar{x}$ ranging over sorts from L and $\epsilon > 0$, there is a formula $\psi(\bar{x}, \bar{z})$ in L such that for every $\bar{b} \in M'$ there is $\bar{c} \in M$ such that

$$\sup_{\bar{x}} |\varphi(\bar{x}, \bar{b}) - \psi(\bar{x}, \bar{c})| \leqslant \epsilon$$

Stable embeddedness, cont'd

Theorem

With T, T' and F, the forgetful functor as above, if F is full then for every $M' \models T'$, $F(M')$ is stably embedded in M'.

- Fix a saturated model M' of T and suppose M is its L-reduct. We'll assume that M is of size continuum and that the continuum hypothesis holds.
- The strategy is to show that if M fails to be stably embedded in M' then we can find an automorphism of M which does not lift to an automorphism of M'.
- To this end, fix a formula $\psi(\bar{x}, \bar{c})$ with $\bar{c} \in M'$ and $\bar{x}$ variables from the sorts of L.
- We wish to defeat all possible places that $\bar{c}$ can go as we construct an automorphism of M.

Stable embeddedness, cont'd

- Key point: suppose that A is a countable subset of M and consider the following set of conditions for a fixed k:

$$\Sigma(A,k) := \{|\varphi(\bar{x}) - \varphi(\bar{y})| \leqslant 1/n : n \in \mathbb{N}, \varphi \in L_A\}$$
$$\cup \{|\psi(\bar{x},\bar{c}) - \psi(\bar{y},\bar{c})| \geqslant 1/k\}$$

- If this set of conditions is not satisfiable for any k then $\psi(\bar{x},\bar{c})$ is a definable predicate over A which contradicts that it is not stably embedded.
- So for each countable A there is a k so that $\Sigma(A,k)$ is consistent.
- We use this to build an automorphism of M which defeats each potential extension to M'.

Another version

Theorem

If F is full and faithful then $F(M')$ is stably embedded in M' for all $M' \models T'$.

- Proof: Fix any model M' of T' and $\psi(\bar{x}, \bar{c})$ as before. Let $M = F(M')$ and consider the set of conditions

$$\Sigma_k := \{|\varphi(\bar{x}) - \varphi(\bar{y})| \leqslant 1/n : n \in \mathbb{N}, \varphi \in L_M\} \\ \cup \{|\psi(\bar{x}, \bar{c}) - \psi(\bar{y}, \bar{c})| \geqslant 1/k\}$$

- As before, if this set of conditions is not satisfiable for all k then ψ is a definable predicate over M which is what we want.
- So assume Σ_k is satisfiable for some k. Fix N' a model of T' and $a, b \in F(N')$ satisfying Σ_k.

Another version, cont'd

- By assumption, a and b have the same type over M.
- By considering a suitable ultrafilter U, we can assume that there is an elementary map $h : F(N') \to F({N'}^U)$ such that h is the identity on M and sends a to b.
- h must arise as the restriction of some $h' : N' \to {N'}^U$ by the fullness of F and h' restricted to M' must be the identity since h is the identity on M and there is a unique lifting of this map to M'.
- So we have $\psi(a, c) = \psi(h'(a), h'(c)) = \psi(b, c)$ which contradicts the choice of a and b.

Imaginaries: the continuous case, canonical parameters

- Fix a complete theory T in a continuous language L and fix a formula $\varphi(\bar{x}, \bar{y})$.
- Consider the formula $\rho_\varphi(\bar{y}, \bar{y}') := \sup_{\bar{x}} |\varphi(\bar{x}, \bar{y}) - \varphi(\bar{x}, \bar{y}')|$.
- ρ_φ defines a pseudo-metric on the product of the sorts corresponding to the $\bar{y}$ variables in all L-structures and $\rho_\varphi(\bar{y}, \bar{y}') = 0$ means $\varphi(\bar{x}, \bar{y})$ and $\varphi(\bar{x}, \bar{y}')$ define the same function of the $\bar{x}$-variables.
- We consider $L_\varphi = L \cup \{S_\varphi, d_\varphi, \pi_\varphi\}$ where S_φ is a new sort, d_φ is its metric symbol and π_φ is a function from the sorts of the $\bar{y}$ variables to S_φ. The uniform continuity modulus for π_φ is the same as the uniform continuity modulus for the $\bar{y}$ variables in φ.

Imaginaries: the continuous case, canonical parameters, cont'd

- If M is a model of T and $X(M)$ is the product of the sorts corresponding to the $\bar{y}$ variables the ρ_φ is a pseudo-metric on $X(M)$. We define an expansion M_φ of M to L_φ by letting $S_\varphi(M_\varphi) = X(M)/\rho_\varphi$ and d_φ is the induced metric; π_φ is the projection from $X(M)$ to $S_\varphi(M_\varphi)$.
- We let $T_\varphi = Th(M_\varphi)$ and again there is a forgetful function from $Mod(T_\varphi)$ to $Mod(T)$. The question is: if N is a model of T_φ and $M = F(N)$ then why is $N \cong M_\varphi$?
- T_φ knows the following information: for all $m, m' \in X(M)$,

$$d_\varphi(\pi_\varphi(m), \pi_\varphi(m')) = \rho_\varphi(m, m')$$

and that π_φ is surjective.

Imaginaries: the continuous case, canonical parameters, cont'd

- These facts guarantee that the map $i : S_\varphi(N) \to X(M)/\rho_\varphi$ given by

$$i(n) = \pi_\varphi^{M_\varphi}(m) \text{ for any } m \in X(M) \text{ such that } \pi_\varphi^N(m) = n$$

  is well-defined and a surjective isometry.
- The elements of the sort S_φ can be thought of as the canonical parameters asociated to the function $\varphi(\bar{x}, \bar{y})$ of the $\bar{x}$ variables when the $\bar{y}$ variables are fixed.

An annoying detail

- It will be necessary to deal with finitely many formulas and their canonical parameters at a time.
- Here is a way to treat this as if there was only one formula:
- Suppose we have formulas $\varphi(\bar{x}, \bar{y}_1), \ldots, \varphi(\bar{x}, \bar{y}_n)$. Consider the formula

$$\psi(\bar{x}, i, \bar{y}_1, \ldots, \bar{y}_n)$$

  where i ranges over some finite ordered index set $a_1 < \ldots < a_n$ and

$$\psi(\bar{x}, i, \bar{y}_1, \ldots, \bar{y}_n) = \varphi(\bar{x}, \bar{y}_j)$$

  when $i = a_j$.
- One checks then that the canonical parameters for ψ range over the union of the canonical parameters for the φ's.

Imaginaries: the continuous case, products

- Fix a complete theory T in a continuous language L.
- Suppose $\bar{S} = \langle S_n : n \in N \rangle$ is a sequence of sorts from L. The goal is to create $\prod_{n \in N} S_n$ as a new sort.
- Take a model of T and let $X_{\bar{S}} = \prod_{n \in N} X_{S_n}(M)$. We need a metric on $X_{\bar{S}}$.
- Suppose d_i is the metric on S_i with bound B_i; let

$$d(\bar{x}, \bar{y}) = \sum_{i \in N} \frac{d_i(x_i, y_i)}{B_i 2^i}$$

  where $\bar{x}, \bar{y} \in X_{\bar{S}}(M)$.
- d is a metric on $X_{\bar{S}}(M)$ which is complete and bounded by 1.
- We have projection maps $\pi_i : X_{\bar{S}}(M) \to X_{S_i}(M)$ sending $\bar{x}$ to x_i.
- Notice that if $d(\bar{x}, \bar{y}) < \delta$ then $d_i(x_i, y_i) < B_i 2^i \delta$ so π_i is uniformly continuous.

Imaginaries: the continuous case, products cont'd

- Let $L_{\bar{S}} = L \cup \{S_{\bar{S}}, d_{\bar{S}}, \{\pi_i : i \in N\}\}$ where $S_{\bar{S}}$ is a new sort, $d_{\bar{S}}$ is its metric symbol and π_i is a function symbol with domain $S_{\bar{S}}$, range S_i and uniform continuity modulus given as above.
- The construction above shows how to take a model M of T and produce a model $M_{\bar{S}}$ of $L_{\bar{S}}$. Let $T_{\bar{S}} = Th(M_{\bar{S}})$.
- Once again we have a forgetful functor $F : Mod(T_{\bar{S}}) \to Mod(T)$ and we would like to see that it is an equivalence of categories.
- We need to see if $N \models T_{\bar{S}}$ and $M = F(N)$ then $M_{\bar{S}} \cong N$ fixing M.

Imaginaries: the continuous case, products cont'd

- For $n \in X_{\bar{S}}(N)$, let $\rho(n) = \langle \pi_i(n) : i \in N \rangle \in \prod_{i \in N} X_{\bar{S}_i}(M)$.
- If this map is a surjective isometry then it commutes with the π_i's and so is an isomorphism.
- Notice that follows from the theory $T_{\bar{S}}$ that for all $n, n' \in X_{\bar{S}}(N)$, and $k \in N$,

$$\left| d_{\bar{S}}(n, n') - \sum_{i \leqslant k} \frac{d_i(\pi_i(n), \pi_i(n'))}{B_i 2^i} \right| \leqslant \frac{1}{2^k}$$

which shows that ρ is an isometry.

Imaginaries: the continuous case, products cont'd

- It is also part of the theory that for any k

$$\sup_{x_1 \in S_1} \ldots \sup_{x_k \in S_k} \inf_{y \in S_{\bar{S}}} \max\{d_i(x_i, \pi_i(y)) : i \leqslant k\}$$

  evaluates to 0.
- By completeness of $X_{\bar{S}}(N)$, ρ is surjective.
- So $M_{\bar{S}} \cong N$ fixing M and $T_{\bar{S}}$ is a conservative extension of T.
- One issue is that the metric we defined is not canonical - there are other metrics we could have used. We will have to return to this.

The Main Theorem

Theorem

- *T is a complete continuous theory in L;*
- *T is contained in T', a complete continuous theory in L' containing L;*
- *the forgetful functor from Mod(T') to Mod(T) is an equivalence of categories, then*
- *every sort in L' is in definable bijection with a definable zero set in L.*

This will tell us by stable embeddedness that every L' function and relation can also be expressed as a definable predicate in L.

A sketch of the proof

- Fix a saturated model M of T' and suppose $c \in S(M)$, S a sort from L'. Consider $\varphi(\bar{x}, c)$ where $\bar{x}$ ranges over sorts from L.
- By stable embeddedness and compactness, for each n, there are $\psi_i(\bar{x}, \bar{y}_i)$ for $i = 1, \ldots, m_n$ such that

$$\min_i \inf_{\bar{y}_i} |\varphi(x, c) - \psi_i(\bar{x}, \bar{y}_i)| \leqslant \frac{1}{2^n}$$

- Let $\bar{\psi}_n$ be the single formula which codes the canonical parameters for $\psi_1 \ldots \psi_{m_n}$ and $S_{\bar{\psi}_n}$ be the sort of those canoncial parameters.

$$\bar{S}_\varphi = \prod_n S_{\bar{\psi}_n}$$

- The definable predicate $\varphi(\bar{x}, c)$ is captured by an element of $\bar{S}$, a sort entirely in T^{eq}.

A sketch of the proof, cont'd

- Consider

$$\begin{aligned}\Sigma_n = \{\sup_{\bar{x}} |\varphi(\bar{x}, c) - \varphi(\bar{x}, c')| &\leqslant \frac{1}{k} : k \in \omega, \bar{x} \in L\} \cup \\ \{d_S(c, c') &\geqslant \frac{1}{n}\}\end{aligned}$$

- Σ_n is inconsistent by assumption for every n so by compactness there are countably many formulas $\varphi_i(\bar{x}, y)$ such that if two elements of S agree on all these formulas then they are equal.
- So there is a definable injection from S into $\prod_i \bar{S}_{\varphi_i}$ and we can identify S with the definable zero set which is the range of this map.