Averages

Theorem

If T is stable, $\varphi(x, y)$ is a formula and $\epsilon > 0$ then there is a number $N = N(\varphi, \epsilon)$ such that if $\langle a_i : i \in \mathbb{N} \rangle$ is an indiscernible sequence and b is a parameter matching the y-variable then if $S = \limsup \varphi(a_i, b)$ then $|\{i : \varphi(a_i, b) < S - \epsilon\}| < N$.

- Proof: Suppose not. By compactness we want to construct an indiscernible sequence $\langle c_i : i \in Q \rangle$ such that for any $r \in R \backslash Q$, the φ-type $p_r = \{\varphi(c_i, y) \leqslant S - \epsilon : i < r\} \cup \{\varphi(c_i, y) \geqslant S - \epsilon/2 : i > r\}$ is satisfiable.
- If we fix any finitely many conditions in p_r we will want the value $\varphi(c_i, y)$ to be low ($\leqslant S - \epsilon$) for N values of i and high ($\geqslant S - \epsilon/2$) for N larger values of i.

Averages, cont'd

- Consider the indiscernible sequence $\langle a_i : i \in \mathbb{N} \rangle$ and parameter b which are counter-examples to the claim for N.
- But then with this sequence sufficiently pruned one can witness N low values and N high values of φ relative to this choice of b.
- The collection of φ-types p_r contradicts stability.

Definability

- We consider the following formula $Avg(N)(r_1, \ldots, r_{2N-1})$:

$$\min_{w \in [2N-1]^N} \max_{i \in w} r_i$$

- The point of using this formula is that if $\langle c_i : i \in N \rangle$ is an indiscernible sequence, b is any element and φ is a formula then if $N = N(\varphi, \epsilon)$ then $Avg(N)(\varphi(c_1, b), \ldots, \varphi(c_{2N-1}, b))$ is within ϵ of $\limsup \varphi(c_i, b)$.
- Now suppose that $p \in S(M)$. Remember that p is definable over M as we said last week say via being finitely determined.
- We will create a Morley sequence $\langle c_i : i \in N \rangle$ in the type of p and use this sequence to define a global definable type extending p. We do this as follows:

Definability, cont'd

- Let c_0 realize p; if we have defined $c_{<n}$ then let c_n realize the definable extension of p to $Mc_{<n}$.
- Since the sequence of c_i's realize definable extensions of p they form a Morley sequence.
- For any formula φ and $\epsilon > 0$, let $N = N(\varphi, \epsilon)$ and consider $d_p^\epsilon \varphi(y) = Avg(N)(\varphi(c_1, y), \ldots, \varphi(c_{2N-1}, y))$.
- Define a global type **p** by the conditions:

$$\varphi(x, b) = \lim_{\epsilon \to 0} d_p^\epsilon \varphi(b)$$

- This type is consistent since any finite approximation of it is satisfied by c_N for large enough N.

Definability, cont'd

- It is also definable by the $d_p^\epsilon \varphi$'s. The limit of these formulas are definable predicates at first defined over the Morley sequence.
- But these formulas are also equivalent to the φ-definitions of p and so are equivalent to definable predicates over M.
- Conclusion: p is definable over M by the formulas $\lim_{\epsilon \to 0} d_p^\epsilon \varphi(y)$.

A proof of the remaining lemma from stability

Lemma

Suppose T is stable and that $p \in S(M)$. The following are equivalent:

1. $p \cup \varphi(x, a)$ *is contained in a definable extension of* p.
2. *If* $q = t(a/M)$ *then* $d_q\varphi \in p$.
3. $p \cup \varphi(x, a)$ *does not divide over* M.

- We know that a definable extension does not divide so 1 implies 3.
- If $p \cup \varphi(x, a)$ does not divide over M then choose a Morley sequence in q which is used to define $d_q\varphi$.
- By assumption this is consistent with p and p is a complete type so it is in p so 3 implies 2.

Proof, cont'd

- To see 2 implies 1, let **p** be the global definable extension of p. Let $\langle a_i : i \in \mathbb{N}\rangle$ be the Morley sequence used to define $d_q\varphi$ and let a_ω be an additional realization of the definable extension of q over the entire Morley sequence.
- Since $d_q\varphi \in p$, we have $\varphi(x, a_\omega) \in \mathbf{p}$. By automorphisms then $p \cup \varphi(x, a)$ is contained in a definable extension of p.

Recognizing elementarity

Theorem

Suppose that K is a class of metric structures in a language L.

1. *K is the class of models of some theory T iff K is closed under ultraproducts, elementary submodels and isomorphisms.*
2. *K is the class of models of a universal theory iff K is closed under ultraproducts, submodels and isomorphisms.*

- Proof: Left to right is clear in both cases. In the other direction in the first case, let $T = Th(K) = \{\varphi : M \models \varphi \text{ for all } M \in K\}$.
- If M is any model of T, consider the elementary diagram of M, $Diag_{el}(M)$. For any finite $\Delta(\bar{m}) \subseteq Diag_{el}(M)$, there must be $M_\Delta \in K$ such that $M_\Delta \models \inf_{\bar{x}} \Delta(\bar{x})$.
- M then embeds in an ultraproduct of the M_Δ's.

Recognizing elementarity, cont'd

- For the second case, let T be the universal theory of K and use the atomic diagram of M.
- It is worth recording that the ultrafilter used here is what is called regular: an ultrafilter U on I of cardinality λ is called regular if there is a family of $\{V_\alpha : \alpha < \lambda\} \subseteq U$ so that for any $i \in I$, $\{\alpha : i \in V_\alpha\}$ is finite.

Corollary

If K is a class of L-structures and $T = Th(K) = \{\varphi : M \models \varphi \text{ for all } M \in K\}$ then any model of T can be elementarily embedded in an ultraproduct of structures from K via a regular ultrafilter.

Definition

Suppose that H is a Hilbert space. We say that A is a bounded (linear) operator on H if it is linear and there is a number B such that for all $x \in H$, $|Ax| \leqslant B|x|$. The infinum of such B's is called the operator norm of A, $\|A\|$. We write $B(H)$ for the set of all bounded operators on H.

Lemma

1. *A linear operator A on H is bounded iff it is continuous iff it is continuous at 0.*
2. *$B(H)$ is a unital complex algebra i.e. $B(H)$ is closed under +, composition, multiplication by scalars from $\mathbb{C}$ and contains the identity map on H.*

Lemma

1. *Suppose that* $\lambda : H \to C$ *is a linear functional. Then there is a unique* $y \in H$ *such that* $\lambda(x) = \langle x, y \rangle$.
2. *For* $A \in B(H)$, *there is a uniquely defined operator* A^* *such that for all* $x, y \in H$,

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

3. *The operation* $*$ *is an involution on* $B(H)$.

Definition

A C*-algebra is an operator-norm closed $*$-subalgebra of $B(H)$.

Examples of C*-algebras

Examples

1. If H is n-dimensional then $M_n(C)$ is a C*-algebra.
2. Suppose that X is a compact subset of R. Let $L^2(X)$ be square-integrable complex functions on X. This is a Hilbert space via the inner product

$$\langle f, g\rangle = \int_X f\bar{g}dx$$

If $C(X)$ is the collection of continuous complex functions on X then for any $f \in C(X)$, we can associate $A_f : C(X) \to C(X)$ where $A_f(g) = fg$. A_f is linear and one can check that $C(X)$ is a C*-algebra: $A_f^* = A_{\bar{f}}$ and $\|A_f\| = \sup_{x \in X} |f(x)|$.

Ultraproducts of C*-algebras

- Suppose that $A_i \subseteq B(H_i)$ are C*-algebras for $i \in I$ and U is an ultrafilter on I. What would it mean to have an ultraproduct of these algebras?
- What would it act on? $H = \prod_{i \in I} H_i / U$, the ultraproduct of the Hilbert spaces which we have already defined.
- We want to consider only bounded operators on H so let's consider the set

$$A = \{\langle a_i : i \in I \rangle \in \prod_{i \in I} A_i : \text{for some } B, \|a_i\| \leqslant B \text{ for all } i \in I\}$$

- For $\bar{x} \in H$ and $\bar{a} \in A$, let $\bar{a}(\bar{x}) = \langle a_i(x_i) : i \in I \rangle / U$.
- This makes sense since the sequence $\bar{a}$ is bounded and is well-defined since H is the ultraproduct of the Hilbert spaces H_i. You can check this is linear.
- We let the ultraproduct of the A_i's modulo U be the set of operators on H in A. One checks that this is a C*-algebra: it is easy to check that it is closed under *; for norm-closed I am cheating a little.

Back to the case study

- So we have found a class, C*-algebras, that is closed under ultraproducts and subalgebras (use the same Hilbert space and make sure you are a norm-closed *-algebra).
- So (!) C*-algebras should be captured by continous model theory - how?
- Some of the issues here are old: the metric from the operator norm is unbounded and so we will have to consider operator-norm balls of fixed radius as sorts and piece the algebra together. Once we do that though all issues of uniform continuity of $+$, $\times$ and scalar multiplication disappear. * is uniformly continuous no matter what we do.
- In the case of C*-algebras it is possible to include additional sorts for the Hilbert space being acted on. This isn't necessary for two reasons:

A case study, cont'd

- First, we can recover a Hilbert space from an algebraic characterization of C*-algebras due to Gel'fand and Naimark which is useful in its own right.
- Second, adding the Hilbert space doesn't generalize to other contexts notably von Neumann algebras.
- Let's try to capture C*-algebras axiomatically in continuous model theory:
- We introduce sorts B_n for the operator-norm ball of radius n for each $n \in \mathbb{N}$.
- As with Hilbert space, we introduce sorted versions of $+$, $\times$, scalar multiplication and $*$ which do appropriate things when restricted to the sorts.
- Our C*-algebras will be unital and so there will also be a 1 and 0 both in B_1.

Axioms for C*-algebras

- $x + (y + z) = (x + y) + z$, $x + 0 = x$, $x + (-x) = 0$ (where $-x$ is the scalar -1 acting on x), $x + y = y + x$, $\lambda(\mu x) = (\lambda\mu)x$, $\lambda(x + y) = \lambda x + \lambda y$, $(\lambda + \mu)x = \lambda x + \mu x$.
- $1x = x$, $x(yz) = (xy)z$, $\lambda(xy) = (\lambda x)y = x(\lambda y)$, $x(y + z) = xy + xz$;
- $(x^*)^* = x$, $(x + y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$
- $d(x, y) = d(x - y, 0)$; we write $\|x\|$ for $d(x, 0)$.
- $\|xy\| \leqslant \|x\|\|y\|$
- $\|\lambda x\| = |\lambda|\|x\|$
- $\|x^* x\| = \|x\|^2$
- $\sup_{a \in B_1} \|a\| \leqslant 1$

Consequence of the axioms

- The first set of axioms say that any model is a C-vector space.
- The second group guarantee that any model is an algebra.
- The third and fourth items make sure that it is a *-algebra.
- Most of the axioms involving the norm guarantee that we have a normed linear space (note that the relationship with the metric guarantees the triangle inequality).
- $\|x^*x\| = \|x\|^2$ is the so-called C*-equality and one verifies that this holds in the concrete representation of C*-algebras as defined.
- The last axiom goes partway to guaranteeing that the unit ball has the correct meaning; notice that multiplication by N helps determine the N-ball.

Some operator algebra background

- We'll call a complex unital Banach algebra with an involution $*$ satisfying the C*-identity an abstract C*-algebra.
- For any $a \in A$, A an abstract C*-algebra we define $sp(a) = \{\lambda : \lambda 1_A - a \text{ is not invertible}\}$.
- If A is an abstract C*-algebra and a is self-adjoint ($a^* = a$) then $sp(a)$ is a compact subset of $\mathbb{R}$.

Some operator algebra background, cont'd

Theorem (Spectral Theorem)

Suppose that A is an abstract C-algebra and $a \in A$ is self-adjoint. Then the abstract C*-subalgebra $C^*(a)$ generated by a and the identity on A is isomorphic to $C(sp(a))$ via an isomorphism sending a to $id_{sp(a)}$ and id_A to the constant function 1.*

Theorem (Gel'fand-Naimark)

Any abstract C-algebra A is *-isomorphic to a C*-algebra of operators on a Hilbert space.*

Correctness of the axioms

- We now need to show that if we have any model of our axioms then we determine a C*-algebra uniquely up to isomorphism.
- The Gel'fand-Naimark theorem tells us that if we reconstruct the algebra out of the sorts B_n then we have a C*-algebra of operators on a Hilbert space.
- The subtle problem is that we don't know if the sorts B_N are interpreted correctly i.e. is B_N really the operator norm ball of radius N for this algebra.
- We could fix this problem as we did with Hilbert spaces by adding an axiom that makes sure that anything of norm N really is in B_N however this axiom isn't universal and C*-algebras are closed under substructures so this wouldn't be the right axiom.
- Next time we will see how the spectral theorem can save us.