PRESERVING NEAR UNANIMITY TERMS UNDER PRODUCTS

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Abstract. We show that the Maltsev product of two idempotent varieties of algebras that have n-ary and m-ary near unanimity terms, respectively, will have a near unanimity term of arity $n+m-1$. We also show that in general no lower arity near unanimity term can be found.

1. Introduction

This paper deals with general algebraic structures (algebras, for short) and equationally defined classes of algebras, called varieties. Many familiar collections of algebras are varieties, for example, the following classes of algebras can be defined equationally: groups, abelian groups, rings, vector spaces, Boolean algebras. One of the most successful schemes for classifying and organizing the universe of varieties is via the notion of a Maltsev condition, and critical structural properties of varieties often can be correlated with the satisfaction of a particular Maltsev condition.

The prototypical example of a Maltsev condition is the one discovered by Maltsev [12]. He proved that the congruences (kernels of homomorphisms) of algebras in a variety $\mathcal{V}$ permute if and only if $\mathcal{V}$ possesses a ternary term $p(x, y, z)$ such that $\mathcal{V}$ satisfies the equations $p(x, y, y) \approx x$ and $p(x, x, y) \approx y$.

Loosely speaking, a Maltsev condition is a condition that asserts the existence of a set of terms that satisfy a specific set of equations. Since we will be dealing with a particular, special set of Maltsev conditions in this paper, the precise definition will not be provided here. Readers

Date: 26 August 2013.

2010 *Mathematics Subject Classification.* 08B05.

Key words and phrases. Variety, Maltsev product, idempotent, near unanimity term.

The authors were supported by the Natural Sciences and Engineering Research Council of Canada. The second author was also supported by the Arts & Science program at McMaster University.

who wish to learn more about Maltsev conditions, or the basics of universal algebra may consult one of [3], [8], [9], or [11].

Definition 1.1. An **algebra A** consists of a non-empty set A, called the **universe** of **A**, along with a sequence of finitary functions on A, $\langle f_i : i \in I \rangle$, indexed by some set I, called the **basic operations** of **A**. The function $\tau : I \to \mathbb{N}$ that gives the arity, or number of variables, of each of the basic operations f_i is called the **similarity type** of **A**.

Of special significance in this paper are operations and algebras that are idempotent:

Definition 1.2. An operation $f(x_0, x_1, \dots, x_{n-1})$ on a set A is **idempotent** if $f(a, a, \dots, a) = a$ for all $a \in A$. An algebra **A** is idempotent if all of its basic operations are, and a collection of algebras is idempotent if all of its members are.

Two algebras that have the same similarity type share the same basic algebraic language and can be combined together in various ways. For example, just as with groups, one can naturally form the Cartesian product of two similar algebras. In addition, the notions of subgroup, homomorphism, and quotient can be extended to algebras of arbitrary similarity types in a natural way.

A big difference between groups and other types of algebras is that the kernel of a group homomorphism $f : \mathbf{G} \to \mathbf{H}$ is a special type of subgroup of **G**, whereas in general, the kernel of a homomorphism $f : \mathbf{A} \to \mathbf{B}$ is an equivalence relation on A that is compatible with the basic operations of **A**. For groups, this equivalence relation is nothing more than the partition of the group into the cosets of the kernel. As with groups, the collection of kernels of homomorphisms from some algebra **A** forms a lattice.

The basic operations of an algebra **A**, and the symbols used to represent them, can be combined via composition to form derived operations on A, called the term operations and the terms of **A**. For example, one can form the term $p(x, y, z) = x \cdot (y^{-1} \cdot z)$ in the language of groups, and in any given group **G**, one can interpret this term as a ternary operation on G. Note that this term satisfies the conditions for the original Maltsev condition stated earlier.

Definition 1.3. A **variety** is a class $\mathcal{V}$ of algebras, all of the same similarity type, that can be defined by a set of equations involving terms in the language of $\mathcal{V}$.

The Maltsev conditions that we investigate in this paper are those of having a near unanimity term of arity n, for some $n > 2$.

Definition 1.4. Let $\mathbf{A}$ be an algebra and $n > 2$. A term $t(x_0, \ldots, x_{n-1})$ is a **near unanimity term** for $\mathbf{A}$ if $\mathbf{A}$ satisfies the following equations:

$$t(y, x, \ldots, x) \approx t(x, y, x, \ldots, x) \approx \cdots \approx t(x, x, \ldots, x, y) \approx x.$$

We say that $t(x_0, x_1, \ldots, x_{n-1})$ is a near unanimity term for a class of algebras of the same similarity type if it is a near unanimity term for each algebra in the class.

Example 1.5. Consider the two element algebra $\mathbf{B} = \langle \{0, 1\}, \wedge, \vee \rangle$, where 1 and 0 represent the truth values "true" and "false", respectively, and $\wedge$ and $\vee$ are the logical operations "and" and "or". Then the term

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

is a ternary near unanimity term for $\mathbf{B}$.

For a fixed $n > 2$, the condition of having a near unanimity term of arity n is a Maltsev condition that has been well studied [1] and recently has played an important role in developments on the border between algebra and computer science ([10], [2], [14], [4]).

Given two varieties $\mathcal{V}_0$ and $\mathcal{V}_1$ of the same similarity type, one can form their **join**, $\mathcal{V}_0 \vee \mathcal{V}_1$, the smallest variety that contains both $\mathcal{V}_0$ and $\mathcal{V}_1$. If $\mathcal{V}_i$ is generated by an algebra $\mathbf{A}_i$, $i = 0$, 1, then $\mathcal{V}_0 \vee \mathcal{V}_1$ can be seen to be equal to the variety generated by the product $\mathbf{A}_0 \times \mathbf{A}_1$. In this paper, we investigate how well near unanimity terms are preserved under joins of varieties, and more generally under Maltsev products.

Definition 1.6. Let $\mathcal{V}_0$ and $\mathcal{V}_1$ be varieties of the same similarity type τ. The **Maltsev product** of $\mathcal{V}_0$ and $\mathcal{V}_1$, denoted $\mathcal{V}_0 \circ \mathcal{V}_1$, is the class of all algebras $\mathbf{A}$ of similarity type τ such that for some congruence θ of $\mathbf{A}$, the quotient $\mathbf{A}/\theta$ is in $\mathcal{V}_1$ and for each $a \in A$, the θ-class $a/\theta = \{b \in A : (a, b) \in \theta\}$ is a subuniverse of $\mathbf{A}$ such that the subalgebra of $\mathbf{A}$ with universe a/θ belongs to $\mathcal{V}_0$.

Note that when $\mathbf{A}$ is an idempotent algebra, then a/θ will be a subuniverse of $\mathbf{A}$ for any $a \in A$ and congruence θ of $\mathbf{A}$. Also note that in general, $\mathcal{V}_0 \circ \mathcal{V}_1$ will not be a variety, but will always be a quasi-variety (a class defined by implications, rather than equations). In the idempotent case, the Maltsev product can be regarded as a generalization of the usual Cartesian product in that if $\mathbf{B} \in \mathcal{V}_0$ and $\mathbf{C} \in \mathcal{V}_1$ then $\mathbf{B} \times \mathbf{C} \in \mathcal{V}_0 \circ \mathcal{V}_1$ (witnessed by the kernel of the projection onto the second coordinate).

The problem that we resolve in this paper is the following:

Problem 1.7. Suppose that $\mathcal{V}_0$ and $\mathcal{V}_1$ are idempotent varieties of the same similarity type such that $\mathcal{V}_0$ and $\mathcal{V}_1$ have near unanimity terms of arities n and m respectively. Determine the smallest arity of a near unanimity term that $\mathcal{V}_0 \vee \mathcal{V}_1$ and $\mathcal{V}_0 \circ \mathcal{V}_1$ must always have.

We focus on idempotent varieties, since it is known that in general the join of two varieties that have near unanimity terms will not have one [7]. As a warm up exercise, we show that in the idempotent case there will always be a near unanimity term of arity nm.

Proposition 1.8 (Lemma 3.8 of [15] or Corollary 2.6 of [13])**.** *If the idempotent varieties $\mathcal{V}_0$ and $\mathcal{V}_1$ have near unanimity terms of arities n and m respectively, then $\mathcal{V}_0 \circ \mathcal{V}_1$ has a near unanimity term of arity nm.*

Proof. Suppose that $p_0(x_0, x_1, \ldots, x_{n-1})$ and $p_1(x_0, x_1, \ldots, x_{m-1})$ are near unanimity terms for $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively, and consider the nm-ary term $t(x_0, \ldots, x_{mn-1})$:

$$p_0(p_1(x_0, \ldots, x_{m-1}), p_1(x_m, \ldots, x_{2m-1}), \ldots, p_1(x_{m(n-1)}, \ldots, x_{mn-1})).$$

We show that if $\mathbf{A}$ is an algebra and θ is a congruence of $\mathbf{A}$ such that $\mathbf{A}/\theta \in \mathcal{V}_1$ and a/θ is a subuniverse of $\mathbf{A}$ such that the subalgebra of $\mathbf{A}$ with universe a/θ belongs to $\mathcal{V}_0$, then t is a near unanimity term for $\mathbf{A}$. From this we conclude that t is a near unanimity term for $\mathcal{V}_0 \circ \mathcal{V}_1$.

Suppose that $a, b \in A$ and consider the value of $t(a, \ldots, a, b, a, \ldots, a)$ in $\mathbf{A}$. All but one instance of the term p_1 in this evaluation of t, will evaluate to a, since p_1 is an idempotent term operation and so $p_1(a, a, \ldots, a) = a$. The remaining instance of p_1 in t is of the form $c = p_1(a, \ldots, a, b, a, \ldots, a)$ and so in general will not be equal to a. Since p_1 is a near unanimity term for $\mathbf{A}/\theta$, then at least we know that c will be θ-related to a. We then have that this evaluation of t will be equal to $p_0(a, \ldots, a, c, a, \ldots, a)$. Since a and c lie in the same θ-class and p_0 is a near unanimity term for each θ-class, then we can conclude that $p_0(a, \ldots, a, c, a, \ldots, a) = a$, as required. $\square$

In this paper we resolve Problem 1.7 by showing that we can always find a near unanimity term of arity $n + m - 1$ for $\mathcal{V}_0 \vee \mathcal{V}_1$ and $\mathcal{V}_0 \circ \mathcal{V}_1$. We also show that in general one cannot hope to find a lower arity near unanimity term for $\mathcal{V}_0 \vee \mathcal{V}_1$ or $\mathcal{V}_0 \circ \mathcal{V}_1$.

2. An $(n + m - 1)$-ary near unanimity term

For this section, let $\mathcal{V}_0$ and $\mathcal{V}_1$ be idempotent varieties of the same similarity type and let $p_0(x_0, x_1, \ldots, x_{n-1})$ and $p_1(x_0, x_1, \ldots, x_{m-1})$ be near unanimity terms for $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively, for some n and $m > 2$. Let $d = n + m - 1$.

Definition 2.1. Let S be a subset of the variables $\{x_0, x_1, \ldots, x_{d-1}\}$. A term $t(x_0, x_1, \ldots x_{d-1})$ of arity d is a **near unanimity term for** S if

- t is a near unanimity term for the variety $\mathcal{V}_1$, and
- $\mathcal{V}_0 \circ \mathcal{V}_1$ satisfies the equation $t(x, x, \ldots, x, y, x, \ldots, x) \approx x$ whenever y is substituted in t for any one of the variables x_i from S and x is substituted for all of the other variables of t.

A d-ary term that is a near unanimity term for the entire set of d variables will, of course, be a d-ary near unanimity term for the Maltsev product $\mathcal{V}_0 \circ \mathcal{V}_1$.

Theorem 2.2. *If $\mathcal{V}_0$ and $\mathcal{V}_1$ are idempotent varieties of the same similarity type that have n-ary and m-ary near unanimity terms, respectively, then $\mathcal{V}_0 \circ \mathcal{V}_1$ will have a near unanimity term of arity $n+m-1$.*

Proof. Let $p_0(x_0, x_1, \ldots, x_{n-1})$ and $p_1(x_0, x_1, \ldots, x_{m-1})$ be near unanimity terms for $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively, and let $d = n+m-1$. We show by induction on the size of a subset S of the variables $\{x_0, x_1, \ldots, x_{d-1}\}$ that there is d-ary term t_S that is a near unanimity term for S. When $|S| = d$, we will have produced the required term.

The base of our induction is the case $|S| = n$. By suitably permuting variables, we may assume that $S = \{x_0, x_1, \ldots, x_{n-1}\}$. Define t_S to be the term

$$p_0(p_1(x_0, x_n, x_{n+1}, ..., x_{n+m-2}), p_1(x_1, x_n, x_{n+1}, ..., x_{n+m-2}), ..., \\ p_1(x_{n-1}, x_n, x_{n+1}, ..., x_{n+m-2})).$$

It is not hard to see that t_S is a near unanimity term for the variety $\mathcal{V}_1$, since p_1 is and p_0 is idempotent. With a little more effort, it can be verified that $\mathcal{V}_0 \circ \mathcal{V}_1$ will satisfy the equation

$$t_S(x, x, \ldots, x, y, x, \ldots, x) \approx x$$

whenever y is placed in one of the first n variables of t_S. The proof of this is similar to that of Proposition 1.8.

For the induction step, assume that $n-1 < k < d$, that $|S| = k+1$, and that the claim holds for all subsets of variables of size k or less. As in the base case, it suffices to consider the case $S = \{x_0, x_1, \ldots, x_k\}$.

For $0 \le i < n$, let $S_i = \{x_0, x_1, \ldots, x_k\} \setminus \{x_i\}$ and let t_i be a d-ary term that is a near unanimity term for the set S_i. By the induction hypothesis, such terms exist. Define t_S to be the term

$$p_0(t_0(x_0, \ldots, x_{d-1}), t_1(x_0, \ldots, x_{d-1}), ..., t_{n-1}(x_0, \ldots, x_{d-1})).$$

Since each of the terms t_i is a near unanimity term for the variety $\mathcal{V}_1$ and p_0 is an idempotent term, then t_S will also be a near unanimity term for the variety $\mathcal{V}_1$.

To show that t_S is a near unanimity term for the set S, consider the case where y is substituted for x_0 and x is substituted for all of the other variables of t_S. We have that if $\mathbf{A} \in \mathcal{V}_0 \circ \mathcal{V}_1$, witnessed by the congruence θ, then for a, $b \in A$, the element $c = t_0(b, a, \ldots, a) \in a/\theta$, since t_0 is a near unanimity term for the algebra $\mathbf{A}/\theta$. For $i > 0$, $t_i(b, a, \ldots, a) = a$, since t_i is a near unanimity term for the set S_i and $x_0 \in S_i$. So in $\mathbf{A}$,

$$t_S(b, a, a, \ldots, a) = p_0(c, a, a, \ldots, a) = a,$$

since a, $c \in a/\theta$ and p_0 is a near unanimity term for the subalgebra of $\mathbf{A}$ with universe a/θ. The same argument works for any other variable from S and we conclude that t_S is a near unanimity term for S. □

To conclude this section, we note that the depth of the near unanimity term constructed in the proof of Theorem 2.2 is equal to $m + 1$ and that its length, the number of occurrences of p_0 and p_1, is equal to $n^m + n^{m-1} + \cdots + n + 1$. For small values of n and m we have been able to construct slightly shallower and much shorter near unanimity terms [5], and we speculate that with more care, this could be done in general.

3. A lower bound

In this section we show that in general the Maltsev product of two idempotent varieties that have near unanimity terms of arities n and m respectively, will not have a near unanimity term of arity less than $n + m - 1$. In fact, we show that even when considering the Cartesian product of two finite idempotent algebras that each have near unanimity terms, a lower arity cannot be achieved. From this it follows that in the join of the two varieties, a lower arity near unanimity term cannot be found in general.

Let n, $m > 2$ and let τ be the similarity type that consists of n-ary and m-ary operation symbols p_0 and p_1, respectively. We will build two 2-element idempotent algebras $\mathbf{A}_0$ and $\mathbf{A}_1$ of similarity type τ such that for $i = 0$, 1, p_i is a near unanimity term for $\mathbf{A}_i$ and such that the algebra $\mathbf{A}_0 \times \mathbf{A}_1$ does not have a near unanimity term of arity $n + m - 2$. Using Theorem 2.2, we know that this product will have a near unanimity term of arity $n + m - 1$.

Define $\mathbf{A}_0$ to be the algebra of similarity type τ with universe $\{0,1\}$ and basic operations $p_0^{\mathbf{A}_0}$ and $p_1^{\mathbf{A}_0}$ defined by:

$$p_0^{\mathbf{A}_0}(x_0, x_1, \ldots, x_{n-1}) = \bigwedge_{0\le i<j<n} (x_i \vee x_j)$$

$$p_1^{\mathbf{A}_0}(x_0, x_1, \ldots, x_{m-1}) = \bigwedge_{0\le i<m} x_i$$

Define $\mathbf{A}_1$ to be the algebra of similarity type τ with universe $\{0,1\}$ and basic operations $p_0^{\mathbf{A}_1}$ and $p_1^{\mathbf{A}_1}$ defined by:

$$p_0^{\mathbf{A}_1}(x_0, x_1, \ldots, x_{n-1}) = \bigvee_{0\le i<n} x_i$$

$$p_1^{\mathbf{A}_1}(x_0, x_1, \ldots, x_{m-1}) = \bigvee_{0\le i<j<m} (x_i \wedge x_j)$$

It is easy to check that all four of these operations are idempotent and that $p_0^{\mathbf{A}_0}$ and $p_1^{\mathbf{A}_1}$ are near unanimity operations.

In order to show that the product $\mathbf{A}_0 \times \mathbf{A}_1$ does not have a near unanimity term of arity $n+m-2$, we make use of the following proposition.

Proposition 3.1. *Let $k > 2$. If an algebra $\mathbf{A}$ has a k-ary near unanimity term then for all a_i, $b_i \in A$, for $0 \le i < k$, the k-tuple $(a_0, a_1, \ldots, a_{k-1})$ is a member of the subuniverse of $\mathbf{A}^k$ generated by the k-tuples $(b_0, a_1, \ldots, a_{k-1}), (a_0, b_1, \ldots, a_{k-1}), \ldots, (a_0, a_1, \ldots, b_{k-1})$.*

Proof. Suppose that t is a k-ary near unanimity term for $\mathbf{A}$ and let a_i, $b_i \in A$, for $0 \le i < k$. By applying the term t coordinate-wise to the k generators

$$(b_0, a_1, \ldots, a_{k-1}), (a_0, b_1, \ldots, a_{k-1}), \ldots, (a_0, a_1, \ldots, b_{k-1}).$$

and using the fact that t is a near unanimity term on A, we produce the k-tuple $(a_0, a_1, \ldots, a_{k-1})$, as required. □

Theorem 3.2. *The algebra $\mathbf{A}_0 \times \mathbf{A}_1$ does not have a near unanimity term of arity $n+m-2$.*

Proof. We will show that the condition from the previous proposition fails for $k = n+m-2$ and the algebra $\mathbf{A} = \mathbf{A}_0 \times \mathbf{A}_1$. For $0 \le i < m-1$, let $\vec{c}_i$ be the $(n+m-2)$-tuple

$$((0,0), \ldots, (0,0), \mathbf{(0,1)}, (0,0), \ldots, (0,0)\mathbf{;}(1,0), \ldots, (1,0)),$$

where $(0,1)$ occurs in the ith coordinate and $(1,0)$ occurs $n-1$ times. For $0 \le j < n-1$, let $\vec{d}_j$ be the $(n+m-2)$-tuple

$$((0,0), \ldots, (0,0)\mathbf{;}(1,0), \ldots, (1,0), \mathbf{(0,0)}, (1,0), \ldots, (1,0)),$$

where the last occurrence of $(0,0)$ is in coordinate $(m+j-1)$ and $(1,0)$ occurs $n-2$ times. Let S be the subuniverse of $\mathbf{A}^{n+m-2}$ generated by the $\vec{c}_i$ and the $\vec{d}_j$. We will show that the tuple

$$\vec{e} = ((0,0), \ldots, (0,0); (1,0), \ldots, (1,0))$$

is not in S and conclude, by Proposition 3.1, that $\mathbf{A}$ does not have a near unanimity term of arity $(n+m-2)$, as claimed.

Since $\mathbf{A}_0$ and $\mathbf{A}_1$ are idempotent, it is immediate that $\{(0,0),(0,1)\}$ and $\{(0,0),(1,0)\}$ are subuniverses of $\mathbf{A}$. From this it follows that $C = \{(0,0),(0,1)\}^{m-1} \times \{(0,0),(1,0)\}^{n-1}$ is a subuniverse of $\mathbf{A}^{n+m-2}$.

Claim 3.3. *The set $C \setminus \{\vec{e}\}$ is a subuniverse of $\mathbf{A}^{n+m-2}$ that contains the subuniverse S.*

The second part of this claim follows from the first after observing that the generators of S are contained in $C \setminus \{\vec{e}\}$. To prove the first part of the claim, we need to show that $C \setminus \{\vec{e}\}$ is closed under the two basic operations of $\mathbf{A}^{n+m-2}$. That is, if we apply p_i, $i = 0$ or 1, coordinate-wise to elements from $C \setminus \{\vec{e}\}$ then we cannot produce the tuple $\vec{e}$.

The cases $i = 0$ and $i = 1$ are similar and so we will only work through the case $i = 0$. Suppose that we have tuples $\vec{u}_j$ from C, for $0 \le j < n$, with $p_0(\vec{u}_0, \ldots, \vec{u}_{n-1}) = \vec{e}$ in $\mathbf{A}^{n+m-2}$. We will argue that for this to occur, at least one of the $\vec{u}_j$'s must be equal to $\vec{e}$. From this, the claim will follow.

For $0 \le j < n$, since $\vec{u}_j \in C$ then there are elements u_k^j from $\{0,1\}$, for $0 \le k < n+m-2$, with

$$\vec{u_j} = ((0,u_0^j),(0,u_1^j), \ldots, (0,u_{m-2}^j); (u_{m-1}^j, 0), \ldots, (u_{m+n-3}^j, 0)).$$

By examining the first $m-1$ components of $p_0(\vec{u}_0, \ldots, \vec{u}_{n-1}) = \vec{e}$ and referring to the definition of the operation $p_0^{\mathbf{A}_1}$, we can infer that $u_k^j = 0$ for all j and k, with $0 \le j < n$ and $0 \le k < m-1$. We also see that for all k with $m-1 \le k < m+n-2$,

$$p_0^{\mathbf{A}_0}(u_k^0, u_k^1, \ldots, u_k^{n-1}) = 1.$$

For this to occur, for each k, at most one of the u_k^j can equal 0, for $0 \le j < n$. By applying the pigeonhole principle, we conclude that the only way that this can happen is if there is some j with $u_k^j = 1$ for all $m-1 \le k < m+n-2$. For this value of j, we've established that $\vec{u}_j = \vec{e}$, thereby proving the claim.

It is immediate from the claim that the element $\vec{e}$ is not in the subuniverse S and so by Proposition 3.1 the theorem has been proved. □

Corollary 3.4. *For all n, $m > 2$ there are idempotent, finitely generated varieties $\mathcal{V}_0$ and $\mathcal{V}_1$ that have n-ary and m-ary near unanimity terms, respectively, such that $\mathcal{V}_0 \vee \mathcal{V}_1$ and $\mathcal{V}_0 \circ \mathcal{V}_1$ do not have a near unanimity term of arity $n + m - 2$.*

4. Concluding remarks

The results in this paper can be viewed as being part of a wider effort to understand how certain types of Maltsev conditions are preserved under Maltsev products and the joins of varieties. This general study was initiated by Freese and McKenzie in [7].

The first two authors of this paper conducted their research on this project under the supervision of the third author. At the time, Campanella was a Masters student in the graduate program in Mathematics at McMaster University and Conley was an NSERC Undergraduate Student Research Assistant, jointly funded by the Arts & Science program at McMaster. At several key phases in this project, we made use of the Universal Algebra calculator UACalc and the associated java library [6]. We are grateful to Ralph Freese for maintaining this valuable resource.

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