MATH 4LT3/6LT3 Assignment #6

Note: This assignment will not be graded. You are strongly encouraged to work on the following problems and to compare your solutions to the posted solutions. At least one of the following questions will appear on the upcoming final exam. Unless otherwise stated, in your solutions you may use any of the ZFC axioms.

1. (a) Let $\mathcal{E}$ be a set of transitive sets. Show that $\bigcap \mathcal{E}$ is a transitive set.

   (b) Show that every von Neumann ordinal is a transitive set.

   (c) For A a set, define the following sequence of sets: $T_0(A) = \{A\}$, and given $T_n(A)$, $T_{n+1}(A) = \bigcup T_n(A)$. Let $T(A) = \bigcup_{n \geq 0} T_n(A)$.

      i. Explain why the function that send $n \in \mathbb{N}$ to $T_n(A)$ exists.

      ii. Prove that $T(A)$ is a transitive set that contains A.

      iii. Show that if M is any transitive set that contains A then $T(A) \subseteq M$.

   Note that this shows that $T(A)$ is the smallest transitive set that contains A and so is called the transitive closure of A. Another way to show that such a set exists, is to observe that since A is a set, then for some ordinal α, $A \in \mathcal{V}_\alpha$. Since we've shown that this set is transitive, and from part (a) that the intersection of a set of transitive sets is transitive, then the smallest transitive set that contains A is equal to the intersection of all transitive subsets of $\mathcal{V}_\alpha$ that contain A.

   Here is another characterization of $T(A)$. We claim that for a given n, $x \in T_n(A)$ if and only if there are sets $A = x_0 \ni x_1 \ni \cdots \ni x_{n-1} \ni x_n = x$. Call this an $\in$-chain of length n.

2. (a) Let $U = (A, \leq)$ be a best wellordered set. Show that $|A| = ord(U)$. Here, $|A|$ denotes the von Neumann cardinal of the set A.

   (b) Find a well ordered set $V = (B, \leq)$ such that $|B| \neq ord(V)$.

   (c) Show that the class of all von Neumann ordinals, ON, is not a set.

   (d) Show that the class of all von Neumann cardinals, Card_v, is not a set.

3. Recall the definition of $\text{rank}(A)$, the rank of the set A.

(a) What are the ranks of $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$?

(b) Prove that

- if $x \in A$, then $\text{rank}(x) < \text{rank}(A)$.
- $\text{rank}(A) = \sup\{\text{rank}(x) + 1 \mid x \in A\}$.

(c) Show that for α an ordinal, $\text{rank}(\alpha) = \alpha$.

4. Recall the cumulative hierarchy, $\mathcal{V} = \bigcup_{\alpha \in ON} \mathcal{V}_\alpha$. We saw that all sets belong to $\mathcal{V}$.

(a) Show that if $A \in \mathcal{V}_\omega$, then A and $T(A)$ are finite sets and $T(A) \in \mathcal{V}_\omega$. (See question #1.)

(b) Show that for A a set, if $T(A)$ is a finite set, then $A \in \mathcal{V}_\omega$.

(c) Find a finite set B such that $B \notin \mathcal{V}_\omega$.

(d) Show that the set $\mathcal{V}_\omega$ satisfies all of the ZFC axioms, except for the Axiom of Infinity and the Replacement Axiom.

Note that the members of $\mathcal{V}_\omega$ are called the hereditarily finite sets.

5. Consider the following sequence, indexed by the ordinals, of cardinals:

$$\begin{aligned} \beth_0 &= \aleph_0 = |\mathbb{N}|, \\ \beth_{\beta+1} &= 2^{\beth_\beta}, \\ \beth_\lambda &= \sup\{\beth_\beta \mid \beta < \lambda\}, \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

(a) Justify the existence of this sequence.

(b) Show that $|\mathcal{V}_\omega| = \beth_0$.

(c) Show that for any ordinal α, $|\mathcal{V}_{\omega+\alpha}| = \beth_\alpha$. For the definition of ordinal addition used here, consult Theorem 12.19.