MATH 4LT3/6LT3 Assignment #3 Solutions
Due: Friday, October 27, 11:59pm
Upload your solutions to the Avenue to Learn course website.
Detailed instructions will be provided on the course website.

1. Determine which of the following binary relations are well orderings. Justify your answers.

   (a) The relation $\preceq$ on $\mathbb{N}$, where $n \preceq m$ if and only if (n is even and m is odd) or (n and m are both even or both odd, and $n \leq m$).
   Solution: This is a well order on $\mathbb{N}$. It is not hard to see that it is the sum of the well orders $(E, \leq_E)$ and $(O, \leq_O)$, where E is the set of even numbers, O is the set of odd numbers, and $\leq_E$ and $\leq_O$ are the restrictions of $\leq$ to these two sets.

   (b) Let Σ be the usual set of 26 letters $\{a, b, c, \ldots, z\}$ and let S be the set of finite strings over Σ (so $S = \Sigma^*$). Let $\leq$ on S be the usual alphabetical ordering. So,

   $$\text{aaaa} < \text{add} < \text{addition} < \text{algebra} < \text{b} < \text{set} < \text{theory},$$

   for example.
   Solution: The given relation on S is a linear order (exercise), but it is not a well order, since, for example, the following nonempty set does not have a smallest element with respect to $\leq$:

   $$\{a, ab, aab, aaab, aaaab, \ldots\}.$$

   (c) For $n \in \mathbb{N}^+ = \{n \in \mathbb{N} \mid n > 0\}$, let $n^\#$ be the number of distinct prime factors of n. The relation $\sqsubseteq$ on $\mathbb{N}^+$ defined by $n \sqsubseteq m$ if and only if $n^\# < m^\#$ or ($n^\# = m^\#$ and $n \leq m$).
   Solution: Given two distinct positive integers n and m, it is not hard to see that either $n \sqsubset m$ or $m \sqsubset n$, since either one of them has fewer distinct prime factors than the other, or if not, then one is less than the other. That this relation is a partial order is left to the reader to show. To see that this linear order is a well order, let X be a nonempty set of positive integers. Let $X^\#$ be $\{n^\# \mid n \in X\}$. Then $X^\#$ has a smallest element, say k. The smallest element (with respect to $\leq$) y of X such that $y^\# = k$ will be the smallest element of X with respect to $\sqsubseteq$.

2. In Assignment #2, Question 6, the sum and product of linear orders was defined. It was shown that if $(U, \leq)$ and $(V, \preceq)$ are well orders then so are their sums and products. Denote their sum and product by $(U, \leq) + (V, \preceq)$ and $(U, \leq) \times (V, \preceq)$, respectively.

   Let $(U, \leq)$, $(V, \preceq)$, and $(W, \sqsubseteq)$ be well orders.

   (a) Show that

   $$(U, \leq) \times ((V, \preceq) + (W, \sqsubseteq)) =_o ((U, \leq) \times (V, \preceq)) + ((U, \leq) \times (W, \sqsubseteq)).$$

   Solution:

   We can view the product of two ordered sets U and V to be the ordered set that results from replacing each element $v \in V$ by a copy U_v of U, with each copy of U ordered according to the order given on U, and ordering elements of the copy U_v to be less than the elements of the copy U_w if and only in v is less than w according to the order on V.

   In this question, we may assume that the sets U, V, and W are pairwise disjoint. Then elements of the lefthand order are ordered pairs of the form (u, x) where $x \in V \cup W$, while elements in the righthand order are also ordered pairs of this form. We can establish that not only are these two well ordered sets order isomorphic, but that they are equal. Let $\leq_L$ be the well order on the left and $\leq_R$ be the well order on the right. Then for $u, u' \in U$ and $x, x' \in V \cup W$,

   $$\begin{aligned}(u, x) \leq_L (u', x') \Leftrightarrow & (x \in V, x' \in W) \text{ or } (x, x' \in V \text{ and } x \prec x') \text{ or} \\ & (x, x' \in W \text{ and } x \sqsubset x') \text{ or } (x = x' \text{ and } u < u') \\ \Leftrightarrow & (u, x) \leq_R (u', x').\end{aligned}$$

   (There really isn't much more to this than carefully chasing the definitions.)

   (b) Does the identity

   $$((U, \leq) + (V, \preceq)) \times (W, \sqsubseteq) =_o ((U, \leq) \times (W, \sqsubseteq)) + ((V, \preceq) \times (W, \sqsubseteq))$$

   also hold?

Solution: No. For example, let U be a one element well ordered set and let $\mathbb{N}$ denote the usual well ordering of the natural numbers. Then $U + U$ is a two element well ordered set and so is order isomorphic to $\mathbf{2} = (\{0, 1\}, \leq)$, and $(U+U)\times\mathbb{N}$ is order isomorphic to $\mathbf{2}\times\mathbb{N}$. This is the well order obtained by replacing each natural number n with two copies of it, n_0 and n_1, and ordering them so that $n_0 < n_1$, and $m_i < n_j$ if $m < n$. It is not hard to see that this is order isomorphic to $\mathbb{N}$.

On the other hand, $\mathbb{N} \times (U + U)$ is order isomorphic to $\mathbb{N} \times \mathbf{2}$. This is the well order obtained by stacking one copy of $\mathbb{N}$ on top of another copy. This order is not order isomorphic to $\mathbb{N}$, since it has a limit point, namely the 0 element of the top copy of $\mathbb{N}$, while $\mathbb{N}$ does not have such a point.

3. Consider the usual ordering $\leq$ on $\mathbb{R}$. Show that if X is a nonempty subset of $\mathbb{R}$ such that the restriction of $\leq$ to X is a well ordering, then X must be finite or countably infinite.

   Solution: For each $x \in X$, the interval (x, Sx) of real numbers contains an infinite number of rational numbers. Let q_x be some rational number in this interval. Then the map that sends $x \in X$ to $q_x \in \mathbb{Q}$ is an injection. It follows that X is a countable set. Note that for $x \in X$, Sx denotes the successor of x in X, and so is the smallest element $y \in X$ with $x < y$. Also, we can select q_x to be the smallest rational number in (x, Sx), using the well ordering of $\mathbb{Q}$ provided by Cantor.

4. A quasi-order on a set X is a binary relation $\preceq$ on X that satisfies: $x \preceq x$ for all $x \in X$, and if x, y, $z \in X$ with $x \preceq y$ and $y \preceq z$ then $x \preceq z$. So, any partial order on X is a quasi-order on X.

   (a) Let V be a vector space over some field $\mathbb{F}$ and define $\preceq$ on $\mathcal{P}(V)$ by $A \preceq B$ if $\mathrm{Span}(A) \subseteq \mathrm{Span}(B)$. Show that $\preceq$ is a quasi-order on $\mathcal{P}(V)$. Is it a partial order in general?

   Solution: If $A \preceq B \preceq C$ then $\mathrm{Span}(A) \subseteq \mathrm{Span}(B) \subseteq \mathrm{Span}(C)$, so $\mathrm{Span}(A) \subseteq \mathrm{Span}(C)$, establishing that $A \preceq C$. Clearly $A \preceq A$, so $\preceq$ is a quasi-order. It is not in general a partial order, since anti-symmetry can fail. For example, if A is any subset of V that doesn't contain 0, then $A \preceq S \cup \{0\}$ and $A \cup \{0\} \preceq A$ but these two sets are not equal.

(b) Let $\preceq$ be a quasi-order on the set X an define $\sim$ to be the following binary relation on X: $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$. Show that $\sim$ is an equivalence relation on X.

Solution: Since $a \preceq a$ for $a \in X$, then $a \sim a$. The transitivity of $\sim$ follows from the transitivity of $\preceq$. The symmetry of $\sim$ follows from the symmetric form of its definition. So, $\sim$ is an equivalence relation on X.

(c) For X, $\preceq$ and $\sim$ as in the previous part, and $a \in X$, let $[a/\sim]$ denote the equivalence class of $\sim$ that contains a, and let $[X/\sim]$ be the set $\{[a/\sim] \mid a \in X\}$.

Define the binary relation $\leq$ on $[X/\sim]$ by $[a/\sim] \leq [b/\sim]$ if and only if $a \preceq b$. Show that $\leq$ is a well defined relation on $[X/\sim]$ and that it is a partial order on $[X/\sim]$.

Solution: We need to show that if $[a/\sim] = [a'/\sim]$ and $[b/\sim] = [b'/\sim]$, and if $[a/\sim] \leq [b/\sim]$ then $[a'/\sim/ \leq [b'/\sim]$. We have that $a \sim a'$ and $b \sim b'$ and so $a' \preceq a \preceq b \preceq b'$ from which it follows that $a' \preceq b'$ and so $[a'/\sim] \preceq [b'/\sim]$.

The relation $\leq$ is clearly reflexive since $\preceq$ is. Transitivity also follows immediately from the transitivity of $\preceq$. Suppose that $[a/\sim] \leq [b/\sim]$ and $[b/\sim] \leq [a/\sim]$. Then $a \preceq b$ and $b \preceq a$, so $a \sim b$ and thus $[a/\sim] = [b/\sim]$. This shows that $\leq$ is a partial order on $[X/\sim]$.

5. Continuing with the previous problem, let $\preceq$ be a quasi-order on the set X. Suppose that $\preceq$ also satisfies these two conditions:

- For all $x, y \in X$, either $x \preceq y$ or $y \preceq x$.
- For all subsets A of X, there is some $a \in A$ such that $a \preceq b$ for all $b \in A$.

(a) Show that under these additional assumptions, that the relation $\leq$ on $[X/\sim]$ is a well ordering.

Solution: From the previous question we know that $\leq$ is a partial order. To see that it is linear, let $a, b \in X$. By assumption, either $a \preceq b$ or $b \preceq a$ and so, by definition, either $[a/\sim] \leq [b/\sim]$ or $[b/\sim] \leq [a/\sim]$. Thus $\leq$ is a linear order. Let S be a nonempty subset of $[X/\sim]$ and let $S' = \{a \in X \mid [a/\sim] \in S\}$.

By assumption, there is some $a \in S'$ such that $a \preceq b$ for all $b \in S'$. Then the element $[a/\sim] \in S$ is the smallest element of S, since if $[b/\sim] \in S$, then $b \in S'$ and by the choice of a, $a \preceq b$. It follows that $[a/\sim] \leq [b/\sim]$.

(b) Show that the relation $\leq_o$, restricted to the set $WO(A)$ is a quasi-order that satisfies these two additional conditions (you may make use of results from Chapter 7 for some of this). For A a set, $WO(A)$ is the set of well orderings of subsets of A. So

$$WO(A) = \{(U, \sqsubseteq) \mid U \subseteq A \text{ and } \sqsubseteq \text{ is a well ordering of } U\}.$$

Solution: That $\leq_O$ restricted to $WO(A)$ is a quasi-order on $WO(A)$ follows from Proposition 7.29 of the test. The first condition follows from Theorem 7.31, the comparability of well orders theorem. The second condition follows from Corollary 7.33.

This establishes that on $[WO(A)/\sim]$, the order $[U/\sim] \leq [V/\sim]$ if and only if $U \leq_O V$ is a well order. As shown in the proof of Hartog's Theorem, there is no injection from this set into A.