MATH 4LT3/6LT3 Assignment #1 Solutions
Due: Friday, September 22, 11:59pm
Upload your solutions to the Avenue to Learn course website.
Detailed instructions will be provided on the course website.

1. Let S be a countable set. Show that the set of all finite subsets of S is also a countable set. Argue informally, without referencing the axioms of set theory.

   Solution: We may assume that S is nonempty, since for $S = \emptyset$, the result easily follows. For each $n \in \mathbb{N}$, let S_n be the set of all subsets of S that have size exactly n. Then S_{fin}, the set of all finite subsets of S, is equal to $\bigcup_{n\geq 0} S_n$. By Theorem 2.10, it will suffice to show that each of the sets S_n is countable. From Lemma 2.16 we know that S^n is countable, so by Proposition 2.7 it suffices to show that $S_n \leq_c S^n$, since $S^n \leq_c \mathbb{N}$. To see that $S_n \leq_c S^n$ consider the function $f : S_n \to S^n$ that maps the n-element set $\{a_0, a_1, \ldots, a_{n-1}\}$ to the n-tuple $(a_0, a_1, \ldots, a_{n-1})$, where the elements of this set are listed in increasing order. It is clear that f is an injection, which establishes that $S_n \leq_c S^n \leq_c \mathbb{N}$.

2. Recall that Δ is the set of all infinite binary sequences. Show that $\Delta \times \Delta =_c \Delta$. Use this to show that $\mathbb{C} =_c \mathbb{R} \times \mathbb{R} =_c \mathbb{R}$.

   Solution: It suffices to produce a bijection from $\Delta \times \Delta$ to Δ. Let $g : \Delta \times \Delta \to \Delta$ map the pair of infinite binary sequences

   $$((a_0, a_1, \ldots,), (b_0, b_1, \ldots))$$

   to the infinite binary sequence $(a_0, b_0, a_1, b_1, \ldots)$. It can be seen that g is a bijection by observing that it has an inverse, namely the function that maps an infinite binary sequence $(c_0, c_1, \ldots)$ to the pair

   $$((c_0, c_2, \ldots,), (c_1, c_3, \ldots)).$$

   Thus $\Delta \times \Delta =_c \Delta$.

   Since $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, then (clearly), $\mathbb{C} =_c \mathbb{R} \times \mathbb{R}$. To see that $\mathbb{R} \times \mathbb{R} =_c \mathbb{R}$, use the previous result and the facts that $\mathbb{R} =_c \mathcal{P}(\mathbb{N})$

and $\mathcal{P}(\mathbb{N}) =_c \Delta$ (see the remarks just before Lemma 2.24). It should probably be noted that if we have sets A, B, C, and D such that $A =_c B$ and $C =_c D$ then $A \times C =_c B \times D$ (exercise).

3. Show that $(\mathbb{N} \to \mathbb{N}) =_c \mathcal{P}(\mathbb{N})$. $(\mathbb{N} \to \mathbb{N})$ denotes the set of all functions $f : \mathbb{N} \to \mathbb{N}$.

   Solution: We'll use the Schröder-Bernstein Theorem 2.26. As noted in the previous solution, $\Delta =_c \mathcal{P}(\mathbb{N})$ and that $\Delta = (\mathbb{N} \to \{0, 1\})$. Since $(\mathbb{N} \to \{0, 1\})$ is a subset of $(\mathbb{N} \to \mathbb{N})$ it follows that $\mathcal{P}(\mathbb{N}) \leq_c (\mathbb{N} \to \mathbb{N})$.

   For the other inequality, we first observe that if $f \in (\mathbb{N} \to \mathbb{N})$ then it is determined by the set $\{(a, b) \in \mathbb{N} \times \mathbb{N} : f(a) = b\}$. Since this set is a subset of $\mathbb{N} \times \mathbb{N}$, then it is a member of $\mathcal{P}(\mathbb{N} \times \mathbb{N})$. From this we can conclude that $(\mathbb{N} \to \mathbb{N}) \leq_c \mathcal{P}(\mathbb{N} \times \mathbb{N})$. Finally, since $\mathbb{N} \times \mathbb{N} =_c \mathbb{N}$ we get that $\mathcal{P}(\mathbb{N} \times \mathbb{N}) =_c \mathcal{P}(\mathbb{N})$ (exercise, using that for sets A and B, if $A =_c B$ then $\mathcal{P}(A) =_c \mathcal{P}(B)$), which allows us to conclude that $(\mathbb{N} \to \mathbb{N}) \leq_c \mathcal{P}(\mathbb{N})$.

4. Let X be any set. Show that $\bigcup \mathcal{P}(X) = X$ and that $X \subseteq \mathcal{P}(\bigcup X)$. Under what circumstances will this inclusion be proper?

   Solution: To see that $X \subseteq \bigcup \mathcal{P}(X)$, note that for $x \in X$, $\{x\} \in \mathcal{P}(X)$ and so by the definition of the union of a set (of sets), it follows that $x \in \bigcup \mathcal{P}(X)$. So $X \subseteq \bigcup \mathcal{P}(X)$. Alternatively, note that $X \in \mathcal{P}(X)$, so every element of X will be in $\bigcup \mathcal{P}(X)$. For the other containment, suppose that $x \in \bigcup \mathcal{P}(X)$. Then, by the definition of the union of a set, for some $A \in \mathcal{P}(X)$, $x \in A$. But $A \subseteq X$, so $x \in X$. Thus $\bigcup \mathcal{P}(X) \subseteq X$ and so $\bigcup \mathcal{P}(X) = X$.

   To see that $X \subseteq \mathcal{P}(\bigcup X)$, let $x \in X$. Then every element of x will be a member of $\bigcup X$ and so $x \subseteq \bigcup X$, or $x \in \mathcal{P}(\bigcup X)$. It follows that $X \subseteq \mathcal{P}(\bigcup X)$.

   We claim that this inclusion is an equality exactly when X is of the form $\mathcal{P}(A)$ for some set A. Using the previous result, we know that in this case, $A = \bigcup \mathcal{P}(A) = \bigcup X$ and so $X = \mathcal{P}(A) = \mathcal{P}(\bigcup X)$. On the other hand, if $X = \mathcal{P}(\bigcup X)$, then $X = \mathcal{P}(A)$ where $A = \bigcup X$.

5. Let x, y, u, and v be sets such that $\{x, y\} = \{u, v\}$. Show that at least one of the following holds: ($x = u$ and $y = v$) or ($x = v$ and $y = u$). Clearly indicate the axioms of set theory that you use in your solution.

   Solution: Since the sets $\{x, y\}$ and $\{u, v\}$ are equal, then by the Axiom of Extensionality, the elements x and y belong to $\{u, v\}$. So, either $x = u$ or $x = v$. Let's suppose that $x = u$. Similarly, $y = v$ or $y = u$ must hold. In the former case, we have that the condition ($x = u$ and $y = v$) holds. In the latter case it follows that $x = y = u$ and so $\{x, y\} = \{u\}$. From $\{x, y\} = \{u, v\}$ it follows that $\{u\} = \{u, v\}$ and by the Axiom of Extensionality, that $v = u$. In this case we have that $x = y = v = u$ and so the condition ($x = u$ and $y = v$) holds. If instead, $x = v$, then using a similar argument we can conclude that the condition ($x = v$ and $y = u$) holds.

6. Let A and B be sets. Use the axioms to explain why $C = \{x \cap y \mid x \in A, y \in B\}$ is also a set. Show that $(\bigcup A) \cap (\bigcup B) = \bigcup C$.

   Solution: First note that for $x \in A$ and $y \in B$, $x \cap y$ is a set (using the Separation Axiom) and is a subset of x, which is a subset of $\bigcup A$ (this set exists by the Unionset Axiom). So for $x \in A$ and $y \in B$, $x \cap y$ is a member of $\mathcal{P}(\bigcup A)$ (which exists by the Powerset Axiom). Thus C is a collection of elements of $\mathcal{P}(\bigcup A)$. We can apply the Separation Axiom to show that C is a set:

   $$C = \{u \in \mathcal{P}(\bigcup A) : \exists x \exists y (x \in A \wedge y \in B \wedge u = x \cap y)\}.$$

   To show that $(\bigcup A) \cap (\bigcup B) = \bigcup C$, let a be a set. Then $a \in (\bigcup A) \cap (\bigcup B)$ if and only if there are $x \in A$ and $y \in B$ with $a \in x$ and $a \in y$ (this follows from the definition of the unionset). So, this is if and only if $a \in x \cap y$ for some $x \in A$ and $y \in B$. This is equivalent to a belonging to some member of C, and so is equivalent to $a \in \bigcup C$.

7. Show that there is no set A such that $\mathcal{P}(A) \subseteq A$. In your solution you may only use the axioms that are introduced in Chapter 3 of the textbook.

   Solution: Suppose that such a set A exists. Then every subset X of A is also a member of A, since $X \in \mathcal{P}(A) \subseteq A$. We don't need any axioms to justify this, but of course the Powerset axiom can be used to justify that $\mathcal{P}(A)$ exists. So in particular, $A \in A$, since $A \in \mathcal{P}(A)$. We can use the Separation Axiom to show that $X = \{B \in \mathcal{P}(A) : B \notin B\}$ is a set. Then X is a collection of subsets of A, and so is a subset of A (since each subset of A is also a member of A). Now consider whether or not $X \in X$. We see that $X \in X$ if and only if $X \notin X$, which is a contradiction. So such a set A cannot exist.

8. Most other textbooks on Set Theory have a slightly different formulation of the Axiom of Infinity, based on the notion of an inductive set. A set S is **inductive** if $\emptyset \in S$ and for all $x \in S$, the set $x \cup \{x\} \in S$ as well. The more common version of the Axiom of Infinity states that there exists a set that is inductive.

   (a) Argue that an inductive set is infinite.

   Solution: We show by induction on $n \in \mathbb{N}$ that if I is inductive, then it has at least n elements. To show this, we prove a stronger statement. Define the sequence of sets s_n as follows: $s_0 = \emptyset$, and given s_n, define $s_{n+1} = s_n \cup \{s_n\}$. We claim that for each n, $s_n = \{s_0, s_1, \ldots, s_{n-1}\}$ has n-elements and is a member of I. From this it follows that for $n \neq m$, $s_n \neq s_m$ (using the Axiom of Extensionality), and that I is infinite.

   Fo $n = 0$, the claims about s_0 hold, since $\emptyset \in I$. Suppose that $n \geq 0$ and the claims hold for s_n. Then $s_{n+1} \in I$ is guaranteed by the inductive nature of I. By definition, $s_{n+1} = s_n \cup \{s_n\}$ and so, by induction is equal to $\{s_0, s_1, \ldots, s_{n-1}, s_n\}$. This is an $n+1$-element set, since by induction each s_i for $i \leq n$ has i-elements, and so is different from every s_j with $i \neq j \leq n$. Furthermore, s_{n+1} is not equal to any of its predecessors since it has $n+1$ elements, while the predecessors all have few elements.

   (b) Let C be a set. Show that the collection of all sets $X \subseteq C$ that are inductive is a set. In your solution to this, and the remaining parts of this question, indicate the axioms of set theory that you use to establish it.

   Solution: The property of being inductive is definite, in the sense that it can be defined using a first-order formula $I(x)$, where

$$I(x) = \emptyset \in I \wedge \forall x(x \in I \rightarrow x \cup \{x\} \in I).$$

Note that in this formula we are using abbreviations for the statement $\emptyset \in I$ and for $x \cup \{x\} \in I$. The axioms of extensionality, pairset, unionset, and emptyset are used to show that these abbreviations are valid.

We can use the Separation Axiom and the Powerset Axiom to show that the collection of all inductive subsets of C is a set, since this set is equal to:

$$\{X \in \mathcal{P}(C) : I(X)\}.$$

(c) Let C be a set of inductive sets. Show that $\bigcap C$ is an inductive set.

Solution: We note that the existence of $\bigcap C$ as a set is guaranteed by the Separation Axiom. Since by definition, $\emptyset$ belongs to all inductive sets, then it belongs to each member of C and so is in $\bigcap C$. Now suppose that $x \in \bigcap C$. Then for any $Y \in C$, $x \in Y$ and, since Y is inductive, $x \cup \{x\} \in Y$. So $x \cup \{x\} \in \bigcap C$, which shows that $\bigcap C$ is an inductive set.

(d) Let I be an inductive set. Show that the intersection of the set of all inductive subsets of I is also an inductive set.

Solution: This follows from (b) and (c).

(e) Let N be the set from the previous part. Show that N is a subset of every inductive set X. (This set N can be regarded as a copy of the set of natural numbers in our set theoretic universe.)

Solution: Let X be any inductive set. Then from (c), the intersection $I \cap X$ is also inductive, and is a subset of I. Since N is the intersection of all inductive subsets of I, it is a subset of each of them, and in particular, $N \subseteq I \cap X$. From this it follows that $N \subseteq X$, as required.

9. Show that the collection $\mathcal{T}$ of all 2-element sets is a class by producing a formula $\tau(x)$ in the first order language of set theory such that for A a set, $\tau(A)$ is true if and only if A has exactly two elements. Is $\mathcal{T}$ a set?

   Solution: Let $\tau(x)$ be the following formula (that asserts that x contains exactly two elements):

   $$\exists u \exists v[(u \neq v) \wedge u \in x \wedge v \in x \wedge \forall z(w \in x \to (w = u \vee w = v))].$$

   It follows that $\mathcal{T}$ is a class.

   To see that $\mathcal{T}$ is not a set, suppose that it is. Then by the Separation Axiom,

   $$V_1 = \{x \in \mathcal{T} : \emptyset \in x\}$$

   is also a (nonempty) set. For example, the set $\{\emptyset, \{\emptyset\}\} \in V_1$. By the Unionset Axiom, $V_2 = \bigcup V_1$ is also a set. We claim that every set belongs to V_2. Certainly $\emptyset \in V_2$ since it belongs to every member of V_1. Let a be any nonempty set. Then $\{\emptyset, a\}$ is a two element set and so belongs to $\mathcal{T}$, and in fact belongs to V_1. But then $a \in V_2$. Thus V_2 contains all sets. As shown by Russell's paradox, V_2 can't be a set.

10. Show that the Empty Set Axiom can be derived from the other axioms that are presented in Chapter 3 of the textbook.

    Solution: Let I be a set that is guaranteed to exist by the Axiom of Infinity. By the separation axiom, the following defines a subset of I: $\{x \in I : x \neq x\}$. Clearly this set is empty, which shows that the existence of an empty set follows from these two axioms. The Axiom of Extensionality ensures that there is at most one empty set.