MATH 4LT3/6LT3 Assignment #1

Due: Friday, September 22, 11:59pm

Upload your solutions to the Avenue to Learn course website.

Detailed instructions will be provided on the course website.

Important notes:

- In this course, all work submitted for grading must be your own. Limited collaboration in planning and thinking through solutions to homework problems is allowed, but no collaboration is allowed in writing up solutions. It is permissible to discuss general aspects of the problem sets with other students in the class, but each person should hand in their own copy of the solutions. By general aspects I mean you can say things like, "Did you use the Axiom of Choice for question 1?" Anything more detailed than this is not acceptable.

  Violation of these rules may be grounds for giving no credit for a homework paper and also for serious disciplinary action.

- You may not submit solutions found on the internet or generated using AI tools, such as ChatGPT. Please carefully read over the course announcement and the Senate Policy on Academic Integrity. It is your responsibility to know and understand this policy. If you have any questions about this, please contact Dr. Valeriote.

- In presenting your solutions, I will be looking for well written, comprehensible answers. Please don't shy away from using complete English sentences to explain your work, and please be careful with how you use quantifiers. Every statement you write down should assert something, and should be used somehow to help solve the problem at hand.

- To submit your solutions, you will need to upload to the course Avenue To Learn site, a **single pdf file** that contains all of your solutions to this assignment. No other file formats will be accepted. Free online tools are readily available to convert many common file formats to the pdf format and to merge pdf files. If you need assistance with this, please contact the course TA or Dr. Valeriote before the assignment deadline.

1. Let S be a countable set. Show that the set of all finite subsets of S is also a countable set. Argue informally, without referencing the axioms of set theory.

2. Recall that Δ is the set of all infinite binary sequences. Show that $\Delta \times \Delta =_c \Delta$. Use this to show that $\mathbb{C} =_c \mathbb{R} \times \mathbb{R} =_c \mathbb{R}$.

3. Show that $(\mathbb{N} \to \mathbb{N}) =_c \mathcal{P}(\mathbb{N})$. $(\mathbb{N} \to \mathbb{N})$ denotes the set of all functions $f : \mathbb{N} \to \mathbb{N}$.

4. Let X be any set. Show that $\bigcup \mathcal{P}(X) = X$ and that $X \subseteq \mathcal{P}(\bigcup X)$. Under what circumstances will this inclusion be proper?

5. Let x, y, u, and v be sets such that $\{x, y\} = \{u, v\}$. Show that at least one of the following holds: ($x = u$ and $y = v$) or ($x = v$ and $y = u$). Clearly indicate the axioms of set theory that you use in your solution.

6. Let A and B be sets. Use the axioms to explain why $C = \{x \cap y \mid x \in A, y \in B\}$ is also a set. Show that $(\bigcup A) \cap (\bigcup B) = \bigcup C$.

7. Show that there is no set A such that $\mathcal{P}(A) \subseteq A$. In your solution you may only use the axioms that are introduced in Chapter 3 of the textbook.

8. Most other textbooks on Set Theory have a slightly different formulation of the Axiom of Infinity, based on the notion of an inductive set. A set S is **inductive** if $\emptyset \in S$ and for all $x \in S$, the set $x \cup \{x\} \in S$ as well. The more common version of the Axiom of Infinity states that there exists a set that is inductive.

   (a) Argue that an inductive set is infinite.

   (b) Let C be a set. Show that the collection of all sets $X \subseteq C$ that are inductive is a set. In your solution to this, and the remaining parts of this question, indicate the axioms of set theory that you use to establish it.

   (c) Let C be a set of inductive sets. Show that $\bigcap C$ is an inductive set.

   (d) Let I be an inductive set. Show that the intersection of the set of all inductive subsets of I is also an inductive set.

(e) Let N be the set from the previous part. Show that N is a subset of every inductive set X. (This set N can be regarded as a copy of the set of natural numbers in our set theoretic universe.)

9. Show that the collection $\mathcal{T}$ of all 2-element sets is a class by producing a formula $\tau(x)$ in the first order language of set theory such that for A a set, $\tau(A)$ is true if and only if A has exactly two elements. Is $\mathcal{T}$ a set?

10. Show that the Empty Set Axiom can be derived from the other axioms that are presented in Chapter 3 of the textbook.