

### Question 1 (2 points)

Here is the table for an operation on a four element set  $G$  whose elements are:  $a, b, c, d$ .


$\cdot$	$a$	$b$	$c$	$d$
$a$	$b$	$a$	$d$	$c$
$b$	$a$	$b$	$c$	$d$
$c$	$d$	$c$	$a$	$b$
$d$	$c$	$d$	$b$	$a$

Is this the Cayley table for a group on the set  $\{a,b,c,d\}$ ? Answer "Yes" or "No". Justify your answer to receive full credit.

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### Question 2 (2 points)

Suppose that  $G$  is a group and  $N$  is a normal subgroup of  $G$  such that  $N$  is abelian and the factor group  $G/N$  is abelian. Must  $G$  be abelian? Answer "Yes" or "No".

To receive full credit, justify your answer by providing a proof if you think that the answer is "Yes", or by providing an example that shows that  $G$  need not be abelian if you think that the answer is "No".


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**Question 3** (2 points)

Let  $Q$  be the group of rational numbers, with addition, and let  $Z$  be the group of integers, with addition. Is the factor group  $Q/Z$  isomorphic to  $Q$ ? Answer "Yes" or "No". Justify your answer to receive full credit.

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**Question 4** (2 points)

Suppose that  $G$  is a group that only has a finite number of distinct subgroups. Must  $G$  be a finite group? Answer "Yes" or "No". Justify your answer to receive full credit.

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**Question 5** (5 points)

(a) Express the following permutation as a product of disjoint cycles. Place your answer in the first textbox below.

$$f = (1\ 2\ 3)(4\ 5)(1\ 6\ 7)(1\ 5).$$

(b) Express the following permutation as a product of transpositions. Place your answer in the second textbox below.

$$g = (1\ 4\ 3)(2\ 5\ 1)(6\ 7).$$

(c) Which of  $f$  and  $g$  are even permutations? Place your answer in the third textbox below.

Enter either "f", "g", "f, g", or "none" as your answer, depending on which of these transpositions are even. For example, if you think that just  $g$  is even, then enter "g".

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**Question 6** (4 points)

- (a) Find an element of the group  $S_5$  that has order 6.
- (b) Does this group have elements that have order greater than 6? Justify your answer.



**Question 7** (3 points)

Let  $R$  and  $S$  be rings and let  $f: R \rightarrow S$  be an **onto** ring homomorphism from  $R$  to  $S$ . Let  $I$  be the kernel of  $f$ .

- (a) What theorem can be used to establish that the factor ring  $R/I$  is isomorphic to the ring  $S$ ?
- (b) Give the definition of a map  $g: R/I \rightarrow S$  that is an isomorphism from the factor ring  $R/I$  to the ring  $S$ . You do not need to justify your answer, just give the definition of a function  $g$  with this property.

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**Question 8** (2 points)

Consider the following polynomials from the polynomial ring

$$\mathbb{Z}_7[x] :$$

$$f(x) = -3x^4 + 2x^3 - x^2 + 4x + 5$$

and

$$g(x) = 3x^2 + 2.$$

There are polynomials  $q(x)$  and  $r(x)$  from this ring such that

$$f(x) = q(x)g(x) + r(x)$$

with the degree of  $r(x)$  equal to 0 or to 1, or with  $r(x)$  the zero polynomial. From the following list, select the unique polynomial  $r(x)$  with this property.

Remember that the coefficients are from the ring of integers, modulo 7, so your calculations and answers should take this into account. For example, in that ring, the elements -3 and 4 are equal, and, for instance, the product of the element 3 with the element 4 is the same as the element 5.

$4x + 1$

$0$

$5x + 2$

$3x$

$1$

$2x - 3$

$x - 1$

**Question 9** (4 points)

Consider the following list of rings:

(a)  $\mathbb{Z}$

(b)  $\mathbb{Q}$

(c)  $\mathbb{R}[x]$

(d)  $M_2(\mathbb{R})$  (the  $2 \times 2$  matrices over  $\mathbb{R}$ )

(e)  $3\mathbb{Z}$

(f)  $\mathbb{Z}_{11}$

(g)  $\mathbb{Z}_{12}$

(h)  $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$

In the first textbox, enter all of these rings that are **integral domains**. Use the letters (a) through (h) to refer to particular rings. For example, if you think that the ring of integers modulo 11 is an integral domain, include "(f)" as part of your

answer.

In the second textbox enter all of the rings from this list that are **fields**.

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### Question 10 (4 points)

For each of the following rings  $R$  and subsets  $X$  of  $R$ , determine if  $X$  is an **ideal** of  $R$ :

(#1)  $R = \mathbb{Z}[x],$

$$X = \{p(x) \in \mathbb{Z}[x] \mid p(x) = 0 \text{ or } \deg(p(x)) \leq 2\}$$

(#2)  $R = \mathbb{Z}, X = 5\mathbb{Z}$

(#3)  $R = \mathbb{Q}, X = \mathbb{Z}$

(#4)  $R = \mathbb{R}[x], X = \{p(x) \in \mathbb{R}[x] \mid p(5) = 0\}$

Place your answer ("Yes" if  $X$  is an ideal, and "No" if it isn't) to part (#1) in the first textbox. Place your answers to parts (#2), (#3), and (#4) in the second, third, and fourth textboxes, respectively. You do not need to justify your answers.

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### Question 11 (3 points)

Let  $H$  and  $K$  be normal subgroups of the group  $G$ .

(a) Show that the map

$$f : G \rightarrow G/H \times G/K$$

defined by

$$f(g) = (gH, gK)$$


is a group homomorphism.

(b) Which elements of  $G$  lie in the kernel of  $f$ ? Do not just give the definition of the kernel as your answer.

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### Question 12 (4 points)

Let  $p$  and  $q$  be prime numbers and let  $G$  be a group of order  $pq$ .

(a) What are the possible orders of elements in  $G$  (don't forget to consider the identity element)? Justify your answer.

(b) Is every proper non-trivial subgroup of  $G$  cyclic? Justify your answer.



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**Question 13** (3 points)

Consider the group

$$G = \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5.$$

- (a) List all of the elements of the group  $G$  that have order 3? Place your answer in the first textbox.
- (b) How many elements of order 5 does the group  $G$  have? Place your answer in the second textbox.
- (c) Is the group  $G$  cyclic? Place your answer, "Yes" or "No", in the third textbox.

You do not need to provide justifications for your answers to this question.

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