MATH 3GR3 Assignment #1 Solutions
Due: Friday, September 22, 11:59pm
Upload your solutions to the Avenue to Learn course website.
Detailed instructions will be provided on the course website.

1. Determine which of the following relations are equivalence relations. For those that are, describe the partition that arises from it.

   (a) $R = \{(a, b) \in \mathbb{Z}^2 : ab > 0\}$

   Solution: As presented, R is a binary relation on the set $\mathbb{Z}$. R is not an equivalence relation on the set $\mathbb{Z}$ since the pair $(0, 0)$ is not in R and so R is not reflexive. R is symmetric and transitive and contains (a, a) for all non-zero integers a and so on the set $\mathbb{Z} \setminus \{0\}$, R is an equivalence relation. Two elements of $\mathbb{Z} \setminus \{0\}$ are related by R if and only if they have the same sign and so R partitions this set into two parts, the set of positive integers, and the set of negative integers.

   (b) $R = \{(a, b) \in \mathbb{R}^2 : |a| = |b|\}$

   Solution: R is an equivalence relation on $\mathbb{R}$ since:

   - For all $a \in \mathbb{R}$, $|a| = |a|$ and so $(a, a) \in \mathbb{R}$. (Reflexivity)
   - For all $a, b \in \mathbb{R}$, if $(a, b) \in R$ then $|a| = |b|$ and so $|b| = |a|$, hence $(b, a) \in R$. (Symmetry)
   - For all $a, b, c \in \mathbb{R}$, if $(a, b) \in R$ and $(b, c) \in R$ then $|a| = |b|$ and $|b| = |c|$ and so $|a| = |c|$. Thus $(a, c) \in R$. (Transitivity)

   Two elements of $\mathbb{R}$ are equivalent if they have the same absolute value and so R partitions $\mathbb{R}$ into sets of the form $\{-a, a\}$ for a a positive real number. The number 0 is only related to itself and so $\{0\}$ is the only one-element block of the partition determined by R.

   (c) For $a, b \in \mathbb{N}$, $a \sim b$ if and only if the number of digits in the decimal representations of a and b are the same.

   Solution: The relation $\sim$ is an equivalence relation on $\mathbb{N}$:

   - Reflexivity is automatic, since every $a \in \mathbb{N}$, has a definite decimal (base 10) representation.

- For all $a, b \in \mathbb{N}$, if $a \sim b$ then the number of digits in the base 10 representations of a and b are the same and so $b \sim a$. (Symmetry)
- For all $a, b, c \in \mathbb{N}$, if $a \sim b$ and $b \sim c$ then all three numbers have the same number of digits in their base 10 representations and so $a \sim c$. (Transitivity)

Here are the blocks of the partition determined by $\sim$:

$$\{0, 1, 2, \ldots, 9\}, \{10, 11, 12, \ldots, 99\}, \{100, 101, \ldots, 999\}, \ldots,$$

Note that there are an infinite number of blocks in this partition, one for each positive natural number. Also note that each block is finite. (Question: for $n > 0$, how big is the block consisting of all n-digit numbers?)

(d) For $a, b \in \mathbb{R}$, $a \sim b$ if and only if $|a - b| \leq 2$.

Solution: This relation is not an equivalence relation since it is not transitive (but it is reflexive and symmetric). For example, $1 \sim 2.5$ and $2.5 \sim 4$, but $1 \not\sim 4$.

2. Let $G = \{e, x, y\}$ be any group with three elements. Produce the Cayley (multiplication) table for G.

Solution: The following is the Cayley table for G:

$\cdot$	e	x	y
e	e	x	y
x	x	y	e
y	y	e	x

Justification: The first row and first column values are determined by the equation $e \cdot g = g \cdot e = g$ for all $g \in G$, since e is the identity element. Since each row and column of the table provides a permutation of G then the value of $x \cdot y$ must be either y or e. If $xy = y$ then by cancelation, $e = y$, a contradiction. Thus, $xy = e$. From this it follows that $xx = y$. Similarly, we can show that $yx = e$ and $yy = x$.

3. Let $G = \{e, x, y, z\}$ be a group with four elements, with e the identity element. Show that there are exactly two possibilities for the Cayley

table for G, up to a rearrangement of the elements. This means that any Cayley table that can be obtained from another one by, for example, interchanging y with z in all places in the table, should be considered as the same table.

Solution: As in the previous question, the first row and column of the table are determined since e is the identity element. Let's consider the possible values for xx. Since the second row is a permutation of G then $xx \in \{e, y, z\}$.

Case 1: $xx = e$. Then $xy = z$, since the third column is a permutation of G and so $xy \neq y$. It then follows that $xz = y$. By symmetry, it follows that $yx = z$, $zx = y$. At this point, there are only two possibilities for yy, either e or x and both of them lead to Cayley tables, as indicated below.

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

$\cdot$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	x	e
z	z	y	e	x

Case 2: $xx = y$. Then $xy = z$, $xz = e$, $yx = z$, $zx = e$, $yz = x$, $zz = y$, $yy = e$, and $zy = x$ are all forced. This completes another Cayley table, that is the same as the second table above, up to a rearrangement of the letters.

Case 3: $xx = z$. This case is similar to case 2.

4. Prove that every non-abelian group G has at least six elements, i.e., every group of size 5 or less is abelian.

Solution:

Using the previous two questions and the fact that every group of size 2 or less is abelian, it will suffice to show that every 5 element group G is abelian. Suppose that $G = \{e, x, y, z, w\}$ is a group and that the elements x and y do not commute, i.e., $xy \neq yx$. It follows that xy and yx must take on the values $\{z, w\}$ since, for example, if $xy = e$ then $x^{-1} = y$, implying that $yx = e$ as well. $xy = x$ can be ruled out, since this would imply that $y = e$. Note that this shows that any non-abelian group must have at least 5 elements and so we really don't need to refer to the previous two problems.

So, we may assume that $xy = z$ and $yx = w$. Let's consider xx. Since the second column of the Cayley table of G is a permutation, it follows that xx takes on a value from $\{e, y, z\}$. But $xx \neq z$, since z already appears in the second row of the table. So either $xx = e$ or $xx = y$. If $xx = y$, then we have that $w = yx = (xx)x = x(xx) = xy = z$, a contradiction. So, $xx = e$. Similarly, we can conclude that $yy = e$. From this, we can fill in a few more values of the Cayley table of G, in particular, $zy = w$. But then, $x = xe = x(yy) = (xy)y = zy = w$, a contradiction. Thus $xy \neq yx$ is not possible and so G is abelian.

5. Describe the set of symmetries of a circular disc. In particular, show that this set is infinite.

   Solution: Assume that the disc is centered at the origin and that it has radius 1. Let's label two points by the letters A and B on the perimeter (say the points $(1, 0)$ and $(0, 1)$). A symmetry of the disc will have the effect of moving the point A to some other position on the unit circle. This can be achieved by rotating the disc in a counter clockwise direction by some angle θ with value in the interval $[0, 2\pi)$. The point B must end up either to the left or the right of where the point A was moved. If it is to the left, then the symmetry can be described simply as a rotation of the disc by θ radians.

   On the other hand, if B ends up to the right of A, then this can be achieved by first rotating the disc by θ radians and then by reflecting it about the radial line through the point A. Note that this second sort of symmetry can also be described as a reflection of the disc about some radial line. Of course, since there are an infinite number of angles between 0 and 2π to rotate the disc by, then this symmetry group is infinite.

   Since a symmetry of the disc is completely determined by where it sends the two points A and B, the above describes all of the symmetries of the disc.

6. Describe the set of symmetries of a regular tetrahedron. You do not need to provide the multiplication table for the corresponding group.

Solution: Label the four vertices of the tetrahedron by the letters A, B, C, and D. Then every symmetry of the tetrahedron can be described by some permutation σ of these four letters, since once the positions of the 4 vertices have been determined, the symmetry is also determined. Since there are 24 such permutations, then the symmetry group can have at most 24 elements.

There are a number of ways to see that there are exactly 24 symmetries. First, consider those symmetries that fix vertex A. There are exactly 6 of these: the identity, rotation by $2\pi/3$ radians counter clockwise about the axis through A, rotation by $4\pi/3$ radians, and the reflections though the three planes that contain A along with one of the other vertices. Note that these induce the 6 symmetries of the equilateral triangle with vertices B, C, and D. The remaining 18 symmetries can be obtained by first applying one of these 6 symmetries, and then applying the rotation symmetry that interchanges A with one of the other vertices.

7. Let G be a set and $\circ$ a binary operation on G that satisfies the following properties:

   (a) $\circ$ is associative,

   (b) There is an element $e \in G$ such that $e \circ a = a$ for all $a \in G$,

   (c) For every $a \in G$, there is some $b \in G$ such that $b \circ a = e$.

   Prove that $(G, \circ)$ is a group.

   Solution: We need to show that the element e not only satisfies $e \circ a = a$ for all a, but that $a \circ e = a$. We also need to show that if $b \circ a = e$ then $a \circ b = e$ for all a, $b \in G$. Let $a \in G$ and let $b \in G$ with $ba = e$. From the equality $ee = e$ we get $(ba)e = ba$. We know that there is $c \in G$ with $cb = e$ and so

$$ae = (ea)e = ((cb)a)e = (c(ba))e = c((ba)e) = c(ba) = (cb)a = ea = a.$$

   Now, suppose that $ba = e$. Then $b = eb = (ba)b = b(ab)$. Choose $c \in G$ with $cb = e$. Then

$$e = cb = c(b(ab)) = (cb)(ab) = e(ab) = ab.$$

8. Let G be a group such that $a^2 = e$ for all $a \in G$. Prove that G is abelian.

   Solution: Let $g, h \in G$. Then $gg = hh = (gh)(gh) = e$ since every element of G has order two. From $ghgh = e$ we can obtain $ghghh = eh$ and so $ghg = h$ and from this that $gghg = gh$, which yields $hg = gh$. Thus G is abelian.

9. Let G be a finite group. Show that the number of elements a of G with the property that $a^3 = e$ is odd. Show that the number of elements a of G with the property that $a^2 \neq e$ is even.

   Solution: Note that the element e of G satisfies $e^3 = e$. To show that the set of elements of G with this property is odd, then it will suffice to show that the remaining elements of this set come in pairs. If $a^3 = e$ and $a \neq e$ then $a^{-1} = a^2$ and so $a^{-1} \neq a$. Since $a^3 = e$, then $(a^{-1})^3 = (a^3)^{-1} = e$ and so a^{-1} is in this set. Thus this set consists of the element e and pairs of distinct elements of the form $\{a, a^{-1}\}$ and so has odd size.

   We can use the same sort of argument. Elements of the set $\{a \in G \mid a^2 \neq e\}$ come in pairs of the form $\{a, a^{-1}\}$ and so this set has even size. If $a^2 \neq e$, then it follows that $a^{-2} \neq e$ and so $(a^{-1})^2 \neq e$. Furthermore, $a \neq a^{-1}$ for such an element, or else we would have that $a^2 = aa^{-1} = e$.

10. For the SageMath question, click on the following link to see a solution: link.