Homework 6

1. (a) $f(z) = \dfrac{3}{z^2+2z+1} = \dfrac{3}{(z+1)^2}$. Then $z=-1$ is a double pole of $f(z)$ and

$$\operatorname{Res}\{f(z); -1\} = \frac{d}{dz}(3)\bigg|_{z=-1} = 0$$

(b) $f(z) = \dfrac{1}{z^2\sinh(z)}$. We know that $z = k\pi i$, $k \in \mathbb{Z}$ are zeros of the function $\sinh(z)$ of order 1. Therefore, $z=0$ is a pole of $f(z)$ of order 3. $z = k\pi i$, $k \in \mathbb{Z}\setminus\{0\}$, is the pole of $f(z)$ of order 1 and they are simple covert pole of $f(z)$.

At the point $z=0$, the Laurant expansions of the function $f(z) = \dfrac{1}{z^2\sinh(z)}$ is:

$$f(z) = \frac{1}{z^2\sinh(z)} = \frac{i}{z^2}\cdot\frac{1}{\sin(iz)} = \frac{i}{z^2}\operatorname{cosec}(iz)$$

$$= \frac{i}{z^2}\cdot\frac{1}{iz}\left(1 + \frac{i^2z^2}{3!} + O(z^4)\right)$$

$$= \frac{1}{z^3} - \frac{1}{6z} + O(z)$$

Thus $\operatorname{Res}\{f(z); 0\} = -\frac{1}{6}$

For $k \neq 0$, $\operatorname{Res}\{f(z), k\pi i\} = \frac{1}{z^2} \cdot \frac{1}{\cosh z}\Big|_{z = k\pi i}$

$$= \frac{1}{k^2\pi^2 i^2} \cdot \frac{1}{\cosh(k\pi i)}$$

$$= -\frac{1}{k^2\pi^2} \cdot \frac{2}{e^{k\pi i} + e^{-k\pi i}}$$

$$= -\frac{1}{k^2\pi^2} \cdot \frac{1}{\cos k\pi}$$

$$= \frac{(-1)^{k+1}}{k^2\pi^2}$$

2. proof: Assume that the Laurent expansions of f_1 and f_2 about the point a are the following respectively.

$$f_1(z) = \sum_{n=-m_1}^{\infty} C_n (z-a)^n, \quad f_2(z) = \sum_{n=-m_2}^{\infty} d_n (z-a)^n$$

~~WLOG, we can assume that m~~ , where $m_1, m_2 > 0$

Let $m = \max\{m_1, m_2\}$. Define $C_{-s} = 0$ if $m_1 < s \leq m$ and $d_{-s} = 0$ if $m_2 < s \leq m$.

Then, we have

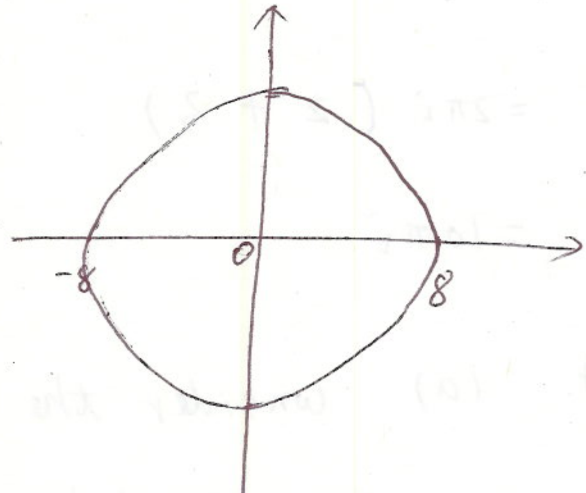
$$f_1(z) = \sum_{n=-m}^{\infty} C_n (z-a)^n \quad \text{and} \quad f_2(z) = \sum_{n=-m}^{\infty} d_n (z-a)^n.$$

By assumption, $c_{-1} = r_1$, $d_{-1} = r_2$

Since $f_1(z) + f_2(z) = \sum\limits_{n=-m}^{\infty} (c_n + d_n)(z-a)^n$, which is the Laurent expansion of $f_1(z) + f_2(z)$ about $z = a$.

Thus, $\operatorname{Res}\{f_1(z) + f_2(z); a\} = c_{-1} + d_{-1} = r_1 + r_2$.

3. (a) $$\int_{\gamma(0;8)} \tan(z)\, dz$$



$$= \int_{\gamma(0;8)} \frac{\sin(z)}{\cos(z)}\, dz$$

Cauchy's residue Theorem

$$= 2\pi i \Big\{ \operatorname{Res}\{f(z); \tfrac{\pi}{2}\} + \operatorname{Res}\{f(z); \tfrac{3\pi}{2}\} + \operatorname{Res}\{f(z); \tfrac{5\pi}{2}\} + \operatorname{Res}\{f(z); -\tfrac{\pi}{2}\} + \operatorname{Res}\{f(z); -\tfrac{3\pi}{2}\} + \operatorname{Res}\{f(z); -\tfrac{5\pi}{2}\} \Big\}$$

~~since~~ $z = \frac{1}{2}(2k+1)\pi$, $k \in \mathbb{Z}$ are the simple covert poles of $f(z) = \tan(z)$. $z = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi$ are inside of $\gamma(0;8)$

$$= 2\pi i \left(\frac{\sin z}{-\sin z} \Big|_{z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}} \right)$$

$$= -12\pi i$$

(b) $\displaystyle\int_{\gamma(0;5)} \frac{5z-2}{z(z-1)}\,dz$

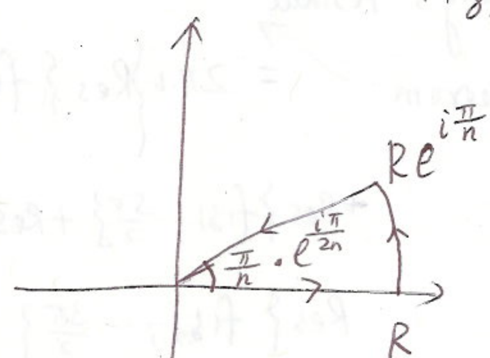
$z=0$ and $z=1$ are the simple overt poles of the function $\dfrac{5z-2}{z(z-1)}$ and $z=0$ and $z=1$ are inside of $\gamma(0;5)$. So Cauchy's residue theorem can be applied.

$$= 2\pi i \left(\mathrm{Res}\left\{ \frac{5z-2}{z(z-1)} ; 0 \right\} + \mathrm{Res}\left\{ \frac{5z-2}{z(z-1)} ; 1 \right\} \right)$$

$$= 2\pi i\, (2+3)$$

$$= 10\pi i$$

4. (a) consider the integral of the function $f(z) = \dfrac{1}{1+z^{2n}}$ on the contour γ shown below



$$\int_\gamma f(z)\,dz = \int_\gamma \frac{1}{1+z^{2n}}\,dz$$

$$= \int_0^R \frac{1}{1+x^{2n}}\,dx + \int_0^{\frac{\pi}{n}} \frac{1}{1+R^{2n}e^{2itn}} \cdot Rie^{it}\,dt + \int_R^0 \frac{e^{i\frac{\pi}{n}}}{1+t^{2n}e^{2i\pi n}}\,dt$$

$$= \int_0^R \frac{1}{1+x^{2n}}\,dx + \int_0^{\frac{\pi}{n}} \frac{Rie^{it}}{1+R^{2n}e^{2int}}\,dt - e^{i\frac{\pi}{n}} \int_0^R \frac{1}{1+t^{2n}}\,dt$$

Note that $z = e^{\frac{i(\pi+2k\pi)}{2n}}$, $k=0,1,\cdots 2n-1$ are the covert simple poles of the function $f(z) = \dfrac{1}{1+z^{2n}}$.

only $z = e^{\frac{i\pi}{2n}}$ is inside of γ. By cauchy's residue theorem, we get.

$$(1-e^{\frac{i\pi}{n}}) \int_0^R \frac{1}{1+x^{2n}}\,dx + \int_0^{\frac{\pi}{n}} \frac{Rie^{it}}{1+R^{2n}e^{2int}}\,dt = 2\pi i \operatorname{Res}\{f(z); e^{\frac{i\pi}{2n}}\}$$

$$= 2\pi i \left.\frac{1}{2n z^{2n-1}}\right|_{z=e^{\frac{i\pi}{2n}}} = \frac{\pi i}{n} e^{(\frac{1}{2n}-1)\pi i}$$

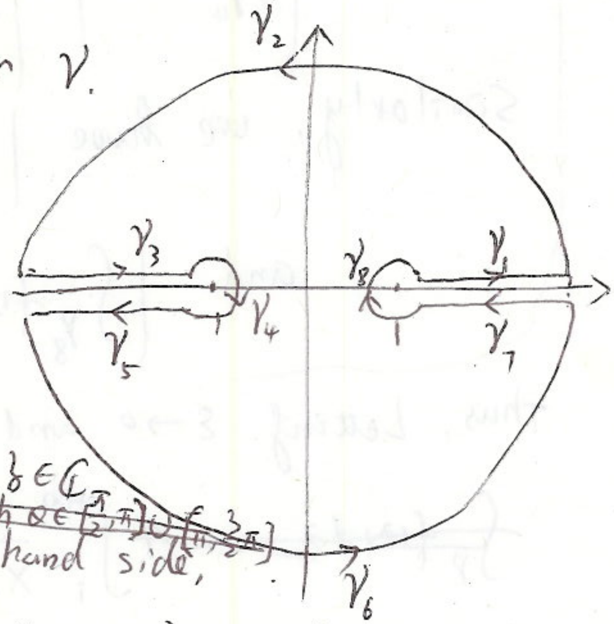
Letting $R \to \infty$, we obtain $\left| \int_0^{\frac{\pi}{n}} \frac{Re^{it} i}{1+R^{2n}e^{2int}}\,dt \right| \to 0$

Thus $\displaystyle\int_0^{+\infty} \frac{1}{1+x^{2n}}\,dx = \frac{\pi i e^{(\frac{1}{2n}-1)\pi i}}{n(1-e^{\frac{i\pi}{n}})}$

Since $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^{2n}}\,dx = 2\int_0^{+\infty} \frac{1}{1+x^{2n}}\,dx$, then

$$\int_{-\infty}^{\infty} \frac{1}{1+x^{2n}}\,dx = \frac{2\pi i e^{(\frac{1}{2n}-1)\pi i}}{n(1-e^{\frac{i\pi}{n}})}$$

(b) Let $f(z) = \dfrac{1}{z\sqrt{z^2-1}}$. We consider the integral of the function f on the following contour γ.



We will control square root of $\sqrt{z^2-1}$ by cutting the plane along $[1,+\infty)$ and along $(-\infty,-1]$. We take the holomorphic branch $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\theta/2}$ for $z \in \mathbb{C}$

~~with $\theta \in (0,\frac{\pi}{2}] \cup [\frac{3}{2}\pi, 2\pi)$ and $\sqrt{z} = |z|^{\frac{1}{2}} e^{i(\theta+2\pi)/2}$ with $\theta \in [\frac{\pi}{2},\pi] \cup [\pi,\frac{3}{2}\pi]$~~

Using the contour shown on the right-hand side,

on γ_1, $\sqrt{z^2-1} = \sqrt{x^2-1}$, ~~$z \in [1+\varepsilon$~~, ~~$z \geq 1$~~ since $z = x \geq 1$ ~~for $z \in \mathbb{C} \cap \gamma_1$~~

$\sqrt{z+1} = \sqrt{x+1}$ and $\sqrt{z-1} = (x-1)^{\frac{1}{2}} e^{i\pi}$

on γ_7, $\sqrt{z^2-1} = -\sqrt{x^2-1}$, where $x \geq 1$ since ~~$z = |z| e^{2\pi i}$ for $z \in \mathbb{C} \cap \gamma_7$~~

For γ_1 and γ_7 we take $\sqrt{z-1} = |z-1|^{\frac{1}{2}} e^{i\arg(z-1)/2}$. For γ_3 and γ_5, we take $\sqrt{z+1} = |z+1|^{\frac{1}{2}} e^{i\arg(z+1)/2}$

Similarly, we have, on γ_3 $\sqrt{z^2-1} = -\sqrt{x^2-1}$ since ~~where~~ $\cancel{z=|z|e^{i\pi}}$

~~and~~ $\cancel{\sqrt{z^2-1} = |z-1|^{\frac{1}{2}}|z+1|^{\frac{1}{2}}e^{\frac{i\pi}{2}}}$ ~~since~~ $\cancel{\sqrt{z^2-1}=|z|}$; on γ_5, $\sqrt{z^2-1} = \cancel{+}\sqrt{x^2-1}$ since $z=|z|e^{i\pi}$

~~and~~ $\cancel{\sqrt{}}$ $\sqrt{z+1} = |z+1|e^{\frac{\pi}{2}i} = (-1-x)^{\frac{1}{2}}i$ and $\sqrt{z-1} = (1-x)^{\frac{1}{2}}i$

for $x \leq -1$. On γ_5, $\sqrt{z^2-1} = \sqrt{x^2-1}$ since $\sqrt{z+1} = |z+1|^{\frac{1}{2}}e^{\frac{3}{2}\pi i}$ $= (-1-x)^{\frac{1}{2}}(-i)$

and $\sqrt{z-1} = (1-x)^{\frac{1}{2}}i$ for $x \leq -1$. Therefore,

$$\int_{\gamma_1} f(z)\,dz = \int_{1+\varepsilon}^{R} \frac{1}{x\sqrt{x^2-1}}\,dx\ ;\quad \int_{\gamma_7} f(z)\,dz = \int_{R}^{1+\varepsilon} \frac{1}{-x\sqrt{x^2-1}}\,dx = \int_{1+\varepsilon}^{R} f(z)\,dz$$

$$\int_{\gamma_5} f(z)\,dz = \int_{-1-\varepsilon}^{-R} \frac{1}{x\sqrt{x^2-1}}\,dx = \int_{1+\varepsilon}^{R} \frac{1}{x\sqrt{x^2-1}}\,dx = \int_{-R}^{-1-\varepsilon} f(z)\,dz = \int_{\gamma_3} f(z)\,dz$$

use the trick in textbook

Furthermore, $\left|\int_{\gamma_2} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{Rie^{it}}{Re^{it}\sqrt{R^2e^{2it}-1}}\,dt\right| \to 0$ as $R \to \infty$

$$\left|\int_{\gamma_4} f(z)\,dz\right| = \left|\int_{-\pi}^{\pi} \frac{\varepsilon i e^{it}}{(-1+\varepsilon e^{it})\sqrt{\varepsilon^2 e^{2it} - 2\varepsilon e^{it}}}\,dt\right| \to 0 \text{ as } \varepsilon \to 0$$

Similarly, we have $\left|\int_{\gamma_6} f(z)\,dz\right| \to 0$ as $R \to \infty$

and $\left|\int_{\gamma_8} f(z)\,dz\right| \to 0$ as $\varepsilon \to 0$.

Thus, Letting $\varepsilon \to 0$ and $R \to \infty$, we get

$$\cancel{\int_{\gamma} f(z)\,dz} = 4\int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = 2\pi, \text{ i.e. } \int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}}\,dx = \frac{\pi}{2}.$$