Homework 5

1. (a) We notice that

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\pi z(1-z^2)}{\sin \pi z} = 1 \neq 0$$

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} \frac{\pi z(1-z)(1+z)}{\sin(\pi(z-1)+\pi)} = \lim_{z\to 1} \frac{\pi z(1-z)(1+z)}{-\sin(\pi(z-1))}$$

$$= 2\pi \neq 0$$

$$\lim_{z\to -1} f(z) = \lim_{z\to -1} \frac{\pi z(1-z)(1+z)}{\sin(\pi(z+1)-\pi)} = \lim_{z\to -1} \frac{\pi z(1-z)(1+z)}{-\sin(\pi(z+1))}$$

$$= -2\pi \neq 0.$$

So the function $f$ has no zeros.

(b) Let $g(z) = \pi z(1-z^2)$ and $h(z) = \sin(\pi z)$. Then $z=0, 1$ and $-1$ are simple zeros of the function $g$ and $z=k$; $k\in\mathbb{Z}$ are simple zeros of the function $h$. Thus $z=k$, $k\in\mathbb{Z}$ are simple poles of the function $\frac{1}{h}$. By 17.13, we obtain that $z=0, 1$ and $-1$ are removable singularities of the function $f=\frac{g}{h}$. and $z=k$, $k\neq 0, 1, -1$ are simple poles of the function $f$.

(c). We know that

$$\frac{1}{\sin \pi z} = \operatorname{cosec} \pi z .$$

$$= \frac{1}{\pi z}\left(1 + \frac{\pi^2 z^2}{3!} + O(\pi^4 z^4)\right) \quad \text{around } 0.$$

Thus the Laurent expansion of $f$ around $0$ is.

$$f(z) = \pi z(1-z^2) \cdot \frac{1}{\pi z}\left(1 + \frac{\pi^2 z^2}{3!} + O(\pi^4 z^4)\right)$$

$$= 1 + \left(\frac{\pi^2}{6} - 1\right) z^2 + O(z^4).$$

2. (a) The Laurent series for $f$ around $0$:

$$f(z) = \frac{1}{z(1-z)(2-z)} = \frac{1}{z} \cdot \frac{1}{1-z} \cdot \frac{1}{2(1-\frac{z}{2})}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} z^n \cdot \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$= \frac{1}{2z} \sum_{n=0}^{\infty} z^n \sum_{n=0}^{\infty} \frac{1}{2^n} z^n$$

$$= \frac{1}{2z}\left(\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} \frac{1}{2^{n-r}}\right) z^n\right)$$

$$= \frac{1}{2z} + \sum_{n=1}^{\infty} \sum_{r=0}^{n} \frac{1}{2^{n+1-r}} z^{n-1}$$

$$= \frac{1}{2z} + \sum_{n=0}^{\infty} \sum_{r=0}^{n+1} \frac{1}{2^{n+2-r}} z^n$$

The coefficent of the principal part is $\frac{1}{2}$

(b). The Laurent series for $f$ around $1$. in the annulus $\{z: 0 < |z-1| < 1\}$

$$f(z) = \frac{1}{z(1-z)(2-z)} = \frac{1}{1+(z-1)} \cdot \frac{1}{1-z} \cdot \frac{1}{1-(z-1)}$$

$$= \frac{1}{1-z} \sum_{n=0}^{\infty} (-1)^n (z-1)^n \sum_{n=0}^{\infty} (z-1)^n$$

$$= -\frac{1}{z-1} \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} (-1)^r \right) (z-1)^n$$

$$= -\frac{1}{z-1} + \sum_{n=1}^{\infty} \sum_{r=0}^{n} (-1)^{r+1} (z-1)^{n-1}$$

$$= -\frac{1}{z-1} + \sum_{n=0}^{\infty} \sum_{r=0}^{n+1} (-1)^{r+1} (z-1)^n$$

The coefficient of the principal part is $-1$.

3. proof: By assumption, there exist $f_1(z)$ and $g_1(z)$ are both holomorphic and $f_1(a) \neq 0$, $g_1(a) \neq 0$ such that $f(z) = (z-a)^k f_1(z)$ and $g(z) = (z-a)^k g_1(z)$.

By 17.13, it's easy to see that $z = a$ is a removable singularity of $\frac{f(z)}{g(z)}$. Then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \lim_{z \to a} \frac{(z-a)^k f_1(z)}{(z-a)^k g_1(z)}$$

$$= \lim_{z \to a} \frac{f_1(z)}{g_1(z)}$$

$$= \frac{f_1(a)}{g_1(a)}$$

$$= \frac{f^{(k)}(a)}{g^{(k)}(a)}.$$

4. (a) $f(z) = \frac{z^2}{\sin^2 z}$ has a removable singularity at $z = 0$ and has covert double poles at $z = k\pi$, $k \neq 0$.

Since $\frac{1}{\sin z} = \operatorname{cosec} z = (-1)^k \operatorname{cosec}(z - k\pi)$

$$= (-1)^k \left[ \frac{1}{z - k\pi} \left( 1 + \frac{(z-k\pi)^2}{3!} + O((z-k\pi)^4) \right) \right]$$

The Laurent series for the function $f(z)$ around $z = k\pi$, $k \neq 0$ is

$$f(z) = \frac{(z - k\pi + k\pi)^2}{\sin^2 z} := \sum_{n=-\infty}^{\infty} C_n (z - k\pi)^n$$

$$= \left[(z-k\pi)^2 + 2k\pi(z-k\pi) + k^2\pi^2\right] \frac{1}{(z-k\pi)^2} \left(1 + \frac{(z-k\pi)^2}{3!} + O((z-k\pi)^4)\right)$$

Then $C_{-1} = 2k\pi$, $k \neq 0$.

Therefore, $\operatorname{Res}\{f(z); k\pi\} = 2k\pi$, $k \neq 0$

$$\operatorname{res}\{f(z); 0\} = 0.$$

(b). $f(z) = \dfrac{z^2-1}{(z^2+1)^2} = \dfrac{z^2-1}{(z+i)^2(z-i)^2}$ has overt double poles at $z = i$ and $z = -i$ respectively.

Then, using 18.8, we have

$$\operatorname{res}\{f(z); i\} = \frac{d}{dz}\left(\frac{z^2-1}{(z+i)^2}\right)\Bigg|_{z=i}$$

$$= \frac{2zi + 2}{(z+i)^3}\Bigg|_{z=i} = 0$$

$$\operatorname{res}\{f(z); -i\} = \frac{d}{dz}\left(\frac{z^2-1}{(z-i)^2}\right)\Bigg|_{z=-i}$$

$$= \frac{-2zi+2}{(z-i)^3}\Bigg|_{z=-i}$$

$$= 0$$